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RENDICONTI

DELLO

VIII CORSO CHE NELLA VILLA MONASTERO A VARENNA

DAL 21 LUGLIO AL 9 AGOSTO 1958

FU TENUTO A CURA

DELLA SCUOLA INTERNAZIONALE DI FISICA

DELLA SOCIETÀ ITALIANA DI FISICA

SUI

*PROBLEMI MATEMATICI DELLA TEORIA QUANTISTICA
DELLE PARTICELLE E DEI CAMPI*

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Parole inaugurali

DI

G. POLVANI

Presidente della Società Italiana di Fisica

Con questa breve cerimonia di stamattina, la quale ricalca le mosse e i movimenti delle due precedenti svolte in occasione dell'inizio dei due Corsi — quello di Fisica del plasma e quello di Teoria dell'informazione — si inizia questo III Corso dell'anno 1958 e VIII della serie totale di tutti quelli curati dalla Scuola Internazionale della Società Italiana di Fisica: esso è relativo, come dice il suo titolo, ai « Problemi matematici della Teoria quantistica delle particelle e dei campi ».

È stata cura del prof. ANTONINO BORSELLINO, di Fisica Teorica dell'Università di Genova, organizzare scientificamente il Corso e ora sarà sua cura il dirigerlo: ed io, conoscendo le fatiche svolte e prevedendo quelle cui si accinge, considerando anche sia la maggior lunghezza che questo Corso ha in confronto degli altri tre di quest'anno, sia anche alcune particolari difficoltà che a questo Corso si connettono, desidero anzi tutto rivolgere a lui il ringraziamento più vivo della Società Italiana di Fisica per il compito assunto.

Egli vi parlerà brevemente, con la sua competenza, delle ragioni per cui questo Corso è sorto e del perchè in Fisica possono alcuni problemi strettamente matematici prendere una così grande importanza da richiedere di essi un particolare studio, e come i risultati che si spera raggiungere per questa via, possano di nuova luce illuminare le questioni fisiche — tutte fondamentali — da cui questi problemi sono alla loro volta scaturiti. Questo Corso metterà dunque l'indice su quelle profonde, essenziali ed intime relazioni tra Matematica e Fisica per le quali parlando di Fisica si fa dell'altissima Matematica e parlando di Matematica si fa dell'altissima Fisica.

Da qui la collaborazione nel Corso tra matematici e fisici, e la presenza tra i partecipanti di puri fisici e di puri matematici. Tutto ciò sembrerebbe costituire un curioso dualismo, rispecchiato anche dal fatto che i quattro cicli in cui il Corso si suddivide si raggruppano anch'essi in due di pura Matematica

e in due di pura Fisica (Analisi funzionale, Teoria dei gruppi e altri formalismi — questioni queste strettamente matematiche —, Caratterizzazione e proprietà delle particelle, Equazioni di evoluzione — questioni queste essenzialmente fisiche —); sembrerebbe, dico, tutto ciò un curioso dualismo, se i due rami di esso non avessero un unico e medesimo scopo: quello di sapere di più e conoscere meglio i modi più intimi e generali di operare della natura e i più sottili e fondamentali di pensare della nostra mente.

Il comune e difficile compito di illustrare queste questioni ponendone in rilievo i risultati raggiunti, le difficoltà che ancora rimangono da superare, le insufficienze di impostazione che ancora forse posseggono le trattazioni, è stato assunto da un gruppo di quattordici docenti che ho l'onore di presentare e il vivo piacere di ringraziare per la collaborazione da loro prestata.

E seguendo una usanza, che ormai si ripete per l'ottava volta in questa nostra Scuola, nominerò anche, presentandoli l'un l'altro e ai maestri, gli allievi e gli uditori di questo Corso per la scelta dei quali ci siamo veramente trovati in grave situazione data la sproporzione enorme tra il numero dei postulanti e quello dei posti disponibili e il profondo disagio tra possibilità e desiderio di accontentare tutti. Ecco dunque i nomi:

Docenti: E. R. CAIANIELLO di Napoli, L. GÅRDING di Lund, M. GOLDHABER di Brookhaven (N.Y.), R. HAAG di Princeton (N.J.), W. HEISENBERG di Göttingen, G. KÄLLÉN di Copenhagen, H. LEHMANN di Hamburg, J. L. LIONS di Nancy, L. MICHEL di Parigi, W. PAULI di Zurigo, G. RACAH di Gerusalemme, B. TOUSCHEK di Roma, J. G. VALATIN di Birmingham, A. S. WIGHTMAN di Princeton (N.J.).

Allievi: Y. AHMAVAARA di Copenhagen, S. ALBERTONI di Milano, D. AMATI di Roma, A. O. BARUT di Syracuse (N.Y.), J. M. BAILEY di Oxford, E. BELLOMO di Genova, J. S. BELL di Harwell, E. BELTRAMETTI di Genova, R. BENGTTSSON di Lund, L. BERTOCCHI di Bologna, M. BOUTEN di Bruges, N. BURGOYNE di Copenhagen, M. CARRASSI di Genova, D. J. CANDLIN di Cambridge, E. CORINALDESI di Stoke-on-Trent (U.K.), G. DELL'ANTONIO di Milano, M. M. DELVES di Oxford, V. DE ALFARO di Torino, G. FANO di Napoli, M. FROISSART di Ginevra, A. FINZI di Roma, S. GASIOROWICZ di Copenhagen, R. GILES di Glasgow, P. GULMANELLI di Milano, J. HILGEVOORD di Amsterdam, G. HÖHLER di München, K. KABIR di Birmingham, C. KACSER di Oxford, D. KASTLER di Marsiglia, N. KHURI di Beirut, J. KNIGHT di College Park, Md., F. KORTEI di Istanbul, G. JONA-LASINIO di Roma, Y. LEHRER di Rehovot, A. LOINGER di Pavia, J. T. LEWIS di Oxford, R. LIOTTA di Roma, G. LUZZATTO di Genova, A. MINGUZZI di Bologna, G. PASSATORE di Genova, G. PATERGNANI di Padova, L. PICASSO di Pisa, C. PIRON di Losanna, R. PRANGE di Philadelphia, T. REGGE di Torino, P.

ROMAN di Manchester, D. RUELLE di Bruxelles, J. RUNDGREN di Stoccolma, R. STORA di Parigi, R. STREATER di Londra, K. SYMANZIK di Gottinga, T. TAKABAYASI di Parigi, L. TENAGLIA di Bari, B. VITALE di Napoli, W. WADA di Washington, G. WANDERS di Gottinga, A. WEIR di Glasgow, K. YAMAZAKI di Gottinga.

Uditori: L. AMERIO di Milano, R. ASCOLI di Torino, N. AUSTERN di Pittsburgh (Pa.), K. BLEULER di Neuchâtel, P. BOCCIERI di Milano, F. DUMIO di Milano, E. MAGENES di Genova, S. NAKAMURA di Ginevra, G. PRODI di Trieste, L. A. RADICATI DI BROZOLO di Pisa, F. ROGER di Bordeaux, G. STAMPACCHIA di Genova, M. VERDE di Torino.

E poichè rifuggo dal pensare che la riconoscenza sia quel sentimento che si prova prima di avere avuto il beneficio, mi si permetta di rinnovare, anche in occasione dell'inizio di questo Corso, le espressioni di viva riconoscenza a tutti coloro — già nominati la volta passata — che hanno dato il loro aiuto finanziario e la loro affettuosa ospitalità, qui a Varenna e in Villa Monastero, a questa nostra Scuola Internazionale di Fisica, che tanto consenso ha ormai acquistato in sede nazionale e internazionale. Essi sentono certo, e con loro ugualmente sentono tutti quelli che partecipano a questa Scuola, che al di fuori e al di sopra di quei vantaggi scientifici immediati che essa produce, un altro vantaggio, su cui tante volte, anzi sempre ho insistito, questa Scuola reca per la sua parte; ed è quello che uomini differenti, per nascita, cultura, tradizione, possono, spinti dal comune desiderio del conoscere e del sapere, avvicinarsi, conoscersi, comprendersi e legarsi tra loro in reciproca stima e in totale spirito di libera convivenza. Condizione questa che oggi più che mai sembra necessaria più che per affermare questa o quella civiltà, per salvare la Civiltà.

Con questi sentimenti, e con l'augurio che ad esso arridano i migliori risultati dichiaro aperto il III Corso 1958, VIII dall'origine della Scuola Internazionale di Fisica della nostra Società.

Prolusione

DI

A. BORSELLINO

Direttore del Corso

La tradizione vuole — ed alle tradizioni pare non ci si possa sottrarre — che il Direttore del Corso debba tenere un discorso al momento della inaugurazione. In tale discorso si vuole che siano illustrate le ragioni che hanno ispirato la istituzione del Corso stesso, le modalità secondo cui si è cercato di realizzarlo, gli scopi che si vogliono raggiungere. Si cerca di fare sia il punto sulle questioni in discussione, sia le previsioni sulle direzioni in cui si stanno sviluppando le ricerche attuali. Il prof. POLVANI ha già toccato in precedenza alcuni di questi aspetti.

Io, iniziando il mio dire, desidero innanzi tutto rivolgere a lui ed alla Società Italiana di Fisica il più vivo ringraziamento per aver accolto con favore la mia proposta di istituire presso la Scuola Internazionale di Varenna, un Corso sopra i « Problemi Matematici della Teoria Quantistica delle Particelle e dei Campi ». Pienamente convinto che un tale Corso dovesse riuscire di notevole utilità, anche se non immediata, il mio piacere per il pronto e partecipe accoglimento della proposta doveva essere fortemente diminuito dal conseguente incarico di curarne la organizzazione. Incarico di non lieve responsabilità, sia per la lunghezza sia per la complessità di un tale Corso, sia infine per la difficoltà intrinseca dell'esperienza che si vuole attuare. Pur conscio dei limiti delle mie capacità, mi son dedicato con passione al compito affidatomi, sostenuto ed incoraggiato dalla fiducia e dalla inesauribilità delle risorse del prof. POLVANI. Ho avuto poi la fortuna di poter contare sull'aiuto del dott. GIUNIO LUZZATTO, dell'Istituto di Fisica di Genova, che fin dall'inizio ha svolto le funzioni di Segretario del Corso con grande capacità e competenza specifica. Anche a lui vada il più sentito ringraziamento per quanto ha fatto e farà ancora durante lo svolgimento del Corso.

Anche se non vogliamo andare troppo indietro nel tempo, è facile constatare nello sviluppo della Fisica periodi di progresso rapidissimo, caratterizzati dal

rigoglioso fiorire di nuove concezioni, di nuove idee, che consentono la sistematizzazione di una grande messe di fatti di esperienza già noti e stimolano verso nuove ricerche. A questi periodi di sviluppo impetuoso seguono talvolta periodi più o meno lunghi, che non possono dirsi di stasi, ma piuttosto di assestamento, di chiarificazione, che preparano il terreno per il successivo balzo avanti. È sufficiente ricordare, per tutti, gli esempi più recenti degli anni intorno al 1900 e quelli intorno al 1925.

Se ci soffermiamo a considerare le due concezioni teoriche più importanti che hanno caratterizzato lo sviluppo della Fisica moderna, la Relatività e la Meccanica quantistica, è immediato osservare che entrambe queste due teorie hanno provocato un notevole allargamento della cultura matematica in normale dotazione di un fisico. La Relatività richiede l'uso del calcolo tensoriale e delle geometrie non euclidee, la Meccanica quantistica quello degli spazi hilbertiani e degli operatori funzionali. Non va sottovalutato il fatto che gli strumenti matematici adeguati per esprimere le nuove idee fisiche erano già stati elaborati in precedenza dai matematici e che erano non eccezionali i matematici al corrente della problematica fisica. Basterà ricordare POINCARÉ e MINKOWSKI da una parte, HILBERT dall'altra. A proposito di quest'ultimo, può essere qui ripetuta la frase da lui pronunciata parlando delle difficoltà che si dovettero superare per dimostrare l'equivalenza della Meccanica delle matrici e della Meccanica ondulatoria: « la Fisica è diventata troppo difficile ("zu schwer") per i fisici ». Espressione forse troppo pessimistica.

La Fisica di questi ultimi trenta anni si è sviluppata essenzialmente intorno a due nuove concezioni, quella delle particelle e quella dei campi, che hanno acquistato una parte fondamentale, ponendo problemi del tutto nuovi, insospettati, ed ancor oggi non del tutto risolti. I problemi matematici che si pongono per chiarire questioni che stanno alla base delle concezioni stesse, appaiono infatti estremamente difficili, e non è certamente semplice distinguere quanto le difficoltà che si incontrano siano di natura puramente matematica o siano alla radice stessa delle concezioni fisiche che si vogliono analizzare. Una delle idee più importanti e feconde, che ha permesso di conservare una notevole fiducia nelle strutture teoriche attuali, nonostante le loro gravi deficienze, l'idea della rinormalizzazione, non può dirsi certo che risolva in modo soddisfacente il problema. Così si auspica che idee fisiche, completamente nuove e rinnovatrici dei fondamenti della teoria, possano sbloccare la situazione, eliminando le attuali difficoltà.

Tuttavia, in qualunque direzione possano condurci queste nuove idee, deve considerarsi molto probabile l'evenienza che esse debbano essere espresse e trattate facendo ricorso ad algoritmi matematici più potenti degli usuali, cioè che si debba far ricorso a strumenti matematici più generali ed astratti. Ma indipendentemente da questa possibilità, si pone il problema, di non trascurabile importanza attuale, al quale ho prima accennato, cioè quello di capire

oggi, nel modo più completo e preciso possibile, quanto le attuali difficoltà che si incontrano siano dovute alla insufficienza della Matematica adoperata e quanto siano intrinseche ai concetti fisici stessi che si studiano. Ad esempio, talune difficoltà formali dovute all'uso delle funzioni improprie di DIRAC, possono essere sistematicamente eliminate facendo ricorso alla teoria delle distribuzioni di SCHWARTZ, che verrà ampiamente esposta nel primo ciclo di questo Corso, quello sull'Analisi funzionale. È indubbio che l'uso sistematico di un algoritmo matematicamente ben fondato, già disponibile sul mercato, in luogo di uno non legalmente riconosciuto (anche se regolarizzabile), può facilitare sia quel chiarimento a cui prima accennavo, sia i contatti, sempre proficui, con i matematici di professione, portandoli ad interessarsi di più delle questioni fisiche moderne.

L'analisi dei rapporti tra Matematica e Fisica è molto interessante sul piano metodologico, ma non è qui il caso di soffermarvisi. Tuttavia a me pare si possa dire che il verificarsi di un distacco, il perdurare di un reciproco disinteresse fra la Matematica moderna e la Fisica teorica possa produrre un danno non trascurabile per lo sviluppo di entrambe le discipline. Ritengo che tutti, nella loro personale esperienza, abbiano constatato le difficoltà gravi, anche di linguaggio, che si incontrano nei contatti tra matematici e fisici, sulle questioni che interessano. Alle difficoltà tecniche si aggiungono spesso incomprensioni, che generano giudizi di valore di uso corrente, inaccettabili anche se possono apparire giustificabili. Si pensi, ad esempio, al diverso atteggiamento che correntemente assumono i fisici nei riguardi dei celebrati problemi esistenziali ai quali i matematici attribuiscono importanza fondamentale. Per un fisico, che ritiene di poter dedurre la prova esistenziale direttamente dalla Natura, tale importanza è uno dei tanti aspetti della inutile pedanteria dei matematici, così privi di senso pratico. E se il matematico spesso si ferma alla sola prova esistenziale, al fisico altrettanto sfugge che quella prova, oltre che assicurare la consistenza della schematizzazione adottata, fornisce a volte anche un metodo costruttivo della soluzione cercata. Mi auguro che mi si vogliano perdonare questi accenni polemici.

Ai fini specifici di questo Corso, ritengo che debba essere considerato un fatto positivo della sua organizzazione, che partecipino ad esso in qualità di docenti o di uditori, dei matematici puri, i quali, per queste tre settimane, potranno rimanere in contatto continuo con dei fisici, scambiare con loro conoscenze, idee, opinioni. Ci auguriamo che tali contatti possano essere proficui a tutti, specie agli allievi.

Del resto, che un Corso di questo genere, qualunque possano essere i suoi effetti immediati o lontani, fosse giustificato e soddisfacesse un bisogno sentito, è provato dal numero assai grande di partecipanti e dal fatto che numerose domande di partecipare al Corso, sono state necessariamente e con nostro vivo dispiacere respinte per necessità funzionali della Scuola.

Dovrei ora illustrare i particolari secondo cui si articola il Corso; ma credo di potermene esimere perchè ritengo che, esaminando il programma dei cicli di lezioni, risultino abbastanza ovvi i criteri che sono stati adottati. Dirò solo che verranno seguiti i due grandi filoni di ricerca delle teorie attuali, la caratterizzazione assiomatica dei campi e quella gruppale delle particelle e le loro connessioni. Ma meglio che dalle mie povere parole, l'organizzazione del Corso si potrà ricavare dalle lezioni dei docenti.

Ai quali desidero rinnovare, anche in questa occasione inaugurale, i ringraziamenti della Scuola sia per la loro partecipazione al Corso, sia anche per i preziosi suggerimenti e consigli dati per l'organizzazione definitiva di esso. Il mio ringraziamento va naturalmente anche a quei professori, che pur avendo in un primo tempo accettato di venire, per diversi motivi non hanno potuto alla fine essere qui presenti. Tra questi desidero ricordare particolarmente i professori KATO e DYSON.

A tutti infine auguro un buon lavoro ed una piacevole permanenza a Varenna.

* * *

In aggiunta alle lezioni ed ai seminari pubblicati in questi rendiconti, durante lo svolgimento del Corso furono tenute altre lezioni e vari seminari i cui testi qui non compaiono: per completezza ne diamo l'elenco.

- L. MICHEL - *Generalities on abstract groups. - Representations of a group; Linear operators. - Representations of compact groups. - Structure of the Lorentz group* (5 lezioni).
- W. HEISENBERG - *The Tamm-Dancoff method. - The Lee model. - A non-linear spinor equation; Group properties* (4 lezioni).
- L. GÅRDING - *The Jost-Lehmann-Dyson representation formula* (1 seminario).
- M. GOLDBABER - *Some problems in weak interactions* (1 seminario).
- J. G. VALATIN - *Field-theoretical methods in the theory of superconductivity* (2 seminari).

L'ultima giornata del Corso è stata dedicata ad una ampia discussione su vari problemi emersi: sotto la Presidenza del compianto Professor W. PAULI, diedero un vivace contributo a questa discussione conclusiva numerosi docenti ed allievi.

SCUOLA INTERNAZIONALE DI FISICA

8° CORSO ESTIVO - VARENNA SUL LAGO DI COMO - VILLA MONASTERO - 21 Luglio - 9 Agosto 1958



A subset A of $M \times M'$ is called a graph.

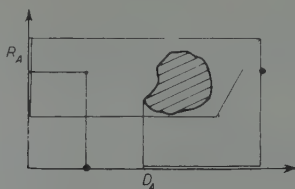


Fig. 2.

The projections of A on M and M' are called the domain $D(A)$ and range $R(A)$ of A respectively. Formally,

$$D(A) = \text{all } \alpha \text{ such that } \{\alpha, \alpha'\} \in A \text{ for at least one } \alpha',$$

$$R(A) = \text{all } \alpha' \text{ such that } \{\alpha, \alpha'\} \in A \text{ for at least one } \alpha.$$

If A and B are graphs and $A \subset B$ we say that A is a restriction of B and B an extension of A . Consider now graphs F with the following property

$$\{\alpha, \alpha'\}, \{\alpha, \beta'\} \in F \Rightarrow \alpha' = \beta'.$$

They are called functions from M to M' (strictly speaking F is a function from $D(F)$ to $R(F)$). The graph of Fig. 2 is no function but the following

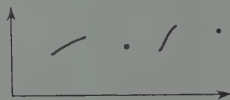


Fig. 3.

Let F be a function. We shall use $\{\alpha, \alpha'\}$, F and $\alpha' = F(\alpha)$ as alternate notations. The last one is traditional. What we have done is to identify symbol F with what is commonly called the graph of a function. Sometimes the elements of M and M' are themselves functions and the terminology may become confusing. In such cases it is customary to use synonyms for the word function, *e.g.* operator, transformation, mapping and, sometimes, functional.

The values $F(\alpha)$ are then referred to as the operator F applied to α or the transform of α or the image of α under the mapping F . Usually, the term functional is applied to the case when M' consists of numbers.

Example. — Let M be all bounded continuous complex-valued functions α defined in an interval $I = (a, b)$. Then, *e.g.*,

$$F(\alpha) = \left| \int_a^b \alpha(x) dx \right|^2,$$

gives a function from M to $M' = \text{the real numbers}$. Let A be the set of sub-intervals J of I . Then $F(\alpha) = \exp \left[\int_J \alpha(x) dx \right]$ is a function F from $A \times M$ to the complex numbers. In both cases the functions are functionals. Next, put

$$(F'(\alpha))(x) = \int_a^x \alpha(t) dt.$$

We get a function (operator, mapping) F' from M to $M' = M$. The range of M is the subset M_0 of M whose elements are continuously differentiable and vanish in a .

Exercise. — How do the orthogonal transformations, the Lorentz transformations, and the Fourier transform fit into the general terminology?

The fundamental operation on functions is composition. (It could be defined also for graphs.) Let F be a function from M to M' and G be a function from M' to M'' . The composition $G \circ F$ of G and F is defined by

$$(G \circ F)(\alpha) = G(F(\alpha)),$$

whenever the right side has a sense. In terms of pairs this means that $G \circ F = \text{all pairs } \{\alpha, \alpha''\} \text{ in } M \times M'' \text{ for which there exists at least one } \alpha' \text{ in } M' \text{ such that } \{\alpha, \alpha'\} \in F \text{ and } \{\alpha', \alpha''\} \in G$. Since F and G are functions, α' and α'' are uniquely determined by α . There is nothing to exclude the possibility that $G \circ F$ is empty. This happens if and only if $R(F) \cap D(G)$ is empty. Let H be a function from M'' to M''' . Then it is easy to convince oneself that

$$H \circ (G \circ F) = (H \circ G) \circ F,$$

i.e. that composition is an associative operation.

Let A be a graph. The inverse graph A^{-1} is defined by an operation of reflection,

$$A^{-1} = \text{all pairs } \{\alpha', \alpha\} \text{ in } M' \times M \text{ such that } \alpha, \alpha' \in A.$$

In particular $(A^{-1})^{-1} = A$ and $R(A^{-1}) = D(A)$ and $D(A^{-1}) = R(A)$. When $A = F$ is a function, F^{-1} may or may not be a function, but if it is, then

$$\begin{aligned} (F^{-1} \circ F)(\alpha) &= \alpha, & \alpha \in D(F), \\ (F \circ F^{-1})(\alpha') &= \alpha', & \alpha' \in R(F). \end{aligned}$$

In other words, if $E = \text{all } \{\alpha, \alpha\}$ and $E' = \text{all } \{\alpha', \alpha'\}$ are the identity functions on M and M' respectively, then $F^{-1} \circ F$ and $F \circ F^{-1}$ are the restrictions of E and E' to $D(F)$ and $R(F)$ respectively.

To illustrate the concept of inverse function let $M = M'$ be the real numbers and put $F(\alpha) = \alpha^2$. Then F^{-1} is the set of pairs $\{\alpha, \pm\sqrt{\alpha}\}$, ($\alpha \geq 0$). It is not a function, but it becomes one by restriction, *e.g.* so that the sign \pm is made a function of α . The obvious choices of taking either $+$ or $-$ for all α are dictated by continuity requirements which so far do not enter into our general framework.

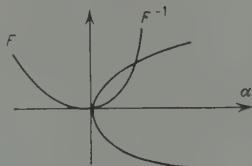


Fig. 4.

Exercise. — Let M be the set of continuous real functions in an interval. Consider the following operators from M to M .

$$(1) \quad F_1(\alpha)(x) = \alpha^3(x) + \alpha(x),$$

$$(2) \quad F_2(\alpha)(x) = \int_a^x \alpha(t) dt,$$

$$(3) \quad F_3(\alpha)(x) = \int_a^x \alpha^2(t) dt.$$

Which are their inverse functions?

The concept of composition very naturally leads to the concept of a group. Let M be a set and consider the set \mathcal{C} of functions F from M to M such that

$$(1) \quad D(F) = R(F) = M,$$

$$(2) \quad F^{-1} \text{ is a function.}$$

Let E be the identity function from M to M . It is immediately verified that \mathcal{C} satisfies the group-axioms: it is closed under composition and the equations

$$F \circ X = G \quad \text{and} \quad Y \circ F = G$$

have the unique solutions $X = F^{-1} \circ G$ and $Y = G \circ F^{-1}$ respectively. When M has n elements $\alpha_1, \dots, \alpha_n$, then \mathcal{C} is nothing but the permutation group of n symbols. In fact, since F^{-1} is a function,

$$F(\alpha_1), \quad \dots, \quad F(\alpha_n)$$

is, for every F in \mathcal{C} , a permutation of $\alpha_1, \dots, \alpha_n$ and every permutation defines a function F in \mathcal{C} . When M is, e.g. the real number the group \mathcal{C} is too large to be of interest, but it contains interesting subgroups, e.g. all F of the form

$$F(\alpha) = a\alpha + b,$$

where $a \neq 0$ and b are real numbers. When M is real n -dimensional space, \mathcal{C} contains e.g. all linear transformations from M to M .

LITERATURE

BOURBAKI: *Théorie des ensembles*.

2. - Linear spaces (L. GÄRDING).

A set L with elements f, g, \dots is called a linear space over the complex numbers C if the following postulates are satisfied.

- 1) There is a function $\{f, g\} \rightarrow f + g$ from $L \times L$ to L such that

$$f + g = g + f, \quad (f + g) + h = f + (g + h).$$

- 2) There is a function $\{a, f\} \rightarrow a \cdot f$ from $C \times L$ to L such that

$$a \cdot (f + g) = a \cdot f + a \cdot g$$

$$(a + b) \cdot f = a \cdot f + b \cdot f$$

$$1 \cdot f = f$$

$$a \cdot (b \cdot f) = (ab) \cdot f.$$

- 3) There is an element 0 of L such that

$$f + 0 = f, \quad a \cdot 0 = 0.$$

Exercises. - Show that the element 0 is uniquely determined and that $-f = -1 \cdot f$ has the property that $f + (-f) = (-f) + f = 0$ for all f . Let f_1, \dots, f_n be elements of L and let L' be all $a_1 \cdot f_1 + \dots + a_n \cdot f_n$, where a_1, \dots are numbers. Then L' is a linear subspace of L of dimension $d \leq n$. If $d = n$ then f_1, \dots, f_n are linearly independent.

In the sequel we write af for $a \cdot f$. The most important examples of a linear space is the set $\mathcal{F}(M, C)$ of all complex-valued functions from a set M to the complex numbers and its linear subspaces. Its elements are all functions $f = f(t)$, $t \in M$, with complex values and $f + g$ and af are defined by

$$(f + g)(t) = f(t) + g(t) \quad \text{and} \quad (af)(t) = af(t)$$

respectively. More generally, let L_0 be a linear space. Then the set $\mathcal{F}(M, L_0)$ of all functions from M to L_0 is a linear space. By choosing a suitable M and suitable subspaces of \mathcal{F} one gets all important linear spaces that occur in practice. We give a list of examples.

1) Let M be the numbers $1, 2, \dots, n$. Then $\mathcal{F}(M, \mathcal{C})$ consists of all vectors $\{\xi_1, \dots, \xi_n\}$ with n complex components.

2) Let \mathcal{O} be an open subset of real n -space and let $\mathcal{E}^k(\mathcal{O})$ be the set of all complex-valued k times continuously differentiable functions from \mathcal{O} to the complex numbers. Then $\mathcal{E}^0(\mathcal{O})$ is a linear space and $\mathcal{E}^k(\mathcal{O})$, ($k > 0$), are decreasing linear subspaces of it. We define the support of a function f defined in \mathcal{O} to be the smallest closed subset outside of which f vanishes. The support is said to be compact (relative to \mathcal{O}) if it is a closed bounded subset of \mathcal{O} . The functions in $\mathcal{E}^k(\mathcal{O})$ with compact supports constitute a linear subspace $\mathcal{D}^k(\mathcal{O})$.

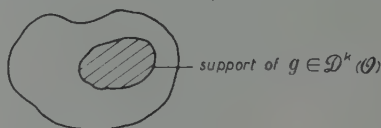


Fig. 5.

Typical functions f in $\mathcal{E}^k(\mathcal{O})$ and g in $\mathcal{D}^k(\mathcal{O})$ are shown in the next figure where \mathcal{O} is an interval.

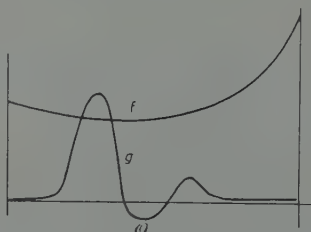


Fig. 6.

Observe that there are no restrictions on the growth of f near the boundary of \mathcal{O} . Imposing restrictions like $f(x) = 0$ when x belongs to some subset of \mathcal{O} , e.g. a finite number of points, we get linear subspaces of $\mathcal{E}^k(\mathcal{O})$ and $\mathcal{D}^k(\mathcal{O})$.

3) The intersections $\mathcal{E}(\mathcal{O}) = \cap \mathcal{E}^k(\mathcal{O})$ and $\mathcal{D}(\mathcal{O}) = \cap \mathcal{D}^k(\mathcal{O})$ are fundamental linear spaces of the theory of distributions. Their elements are infinitely differentiable functions from \mathcal{O} to the complex numbers, those in \mathcal{D} having compact supports.

Exercise. — That \mathcal{D} contains functions besides 0 is not trivial. Let s be a real variable and put

$$\varphi(s) = \exp [-(s+a)^{-1} - (a-s)^{-1}],$$

when $-a < s < a$ and $\varphi(s) = 0$ otherwise. Verify that $\varphi \in \mathcal{D}(R)$, ($R =$ the real line), and construct non-vanishing functions in $\mathcal{D}(\mathcal{O})$.

4) Put $D^j = \partial/\partial x_j$, $D^\alpha = D^{\beta_1} \dots D^{\beta_l}$, $x_\alpha = x_{\alpha_1} \dots x_{\alpha_k}$, $|\beta| = l$, $|\alpha| = k$. Let \mathcal{S}^m be all functions f in $\mathcal{E}^m(R^n)$ for which $x_\alpha D^\beta f(x)$ is bounded for $|\beta| \leq m$ and α arbitrary, (the bound depending on f , α and β). Then \mathcal{S}^m and the intersection \mathcal{S} of $\mathcal{S}^0, \mathcal{S}^1, \dots$, are linear spaces. The space \mathcal{S} is fundamental when it comes to taking Fourier transforms of the tempered distributions.

5) Consider the space H of all functions f from the integers to the complex numbers such that

$$(1) \quad \sum_1^\infty \alpha(j) |f(j)|^2 < \infty,$$

where α is a fixed function from the integers to the positive numbers. Then H is a linear space. We shall see later that it is a Hilbert space. The condition (1) may be replaced by, *e.g.*,

$$\sum \alpha(j) |f(j)| < \infty.$$

We still get a linear space.

Exercise. — Let L and L' be two linear spaces. Their set-theoretical product $L \times L'$ is the set of all pairs,

$$\varphi = \{f, f'\}, \quad f \in L, f' \in L'.$$

Show that if we put

$$\begin{aligned} \varphi + \psi &= \{f + g, f' + g'\} \\ a\varphi &= \{af, af'\}, \end{aligned}$$

then $L \times L'$ becomes a linear space. It is called the direct sum of L and L' and is denoted by

$$L + L'.$$

Exercise. — Let L be a linear space and L_0 a linear subspace. Consider the set \mathcal{M} of all subsets of L of the form

$$\varphi = f + L_0,$$

i.e. translates of L_0 . Show that any two such sets are identical if they have a common element and that this is equivalent to $f - g \in L_0$. Show that if we put

$$\varphi + \psi = f + g + L_0$$

$$a\varphi = af + L_0$$

then M becomes a linear space whose zero element is L_0 . The space M is called the quotient space of L by L_0 and is denoted by

$$L/L_0.$$

Now let L and L' be two linear spaces and consider functions T from L to L' . We say that T is linear or antilinear if

$$T(af + bg) = aT(f) + bT(g)$$

and

$$T(af + bg) = \bar{a}T(f) + \bar{b}T(g)$$

respectively for all complex numbers a and b and all f and g in $D(T)$. When $L' = \mathbb{C}$ the complex numbers, a linear function is commonly called a linear functional. For linear functionals one writes traditionally Tf for $T(f)$.

Exercise. — Show that a subset S of $L \times L'$ is a linear function from L to L' if and only if it is a function and a linear subset of $L + L'$. Show that if T is linear then $D(T)$ and $R(T)$ are linear subspaces of L and L' respectively. Show that if T is linear and T^{-1} is a function then T^{-1} is linear.

Exercise. — Let T and S be two linear functions from all of L into L' . Show that aT and $T + S$ have the same properties. This means that the set $\mathcal{L}(L, L')$ of these functions is a linear space.

Exercise. — Verify the following statements:

- 1) Let D^β be a fixed derivative and h a fixed function in $\mathcal{E}(\mathcal{O})$. Then

$$(Tf)(x) = h D^\beta f(x)$$

defines a linear function T from

- | | | | |
|----|--|----|------------------------------|
| a) | $\mathcal{E}^{k+ \beta }(\mathcal{O})$ | to | $\mathcal{E}^k(\mathcal{O})$ |
| b) | $\mathcal{D}^{k+ \beta }(\mathcal{O})$ | to | $\mathcal{D}^k(\mathcal{O})$ |
| c) | $\mathcal{E}(\mathcal{O})$ | to | $\mathcal{E}(\mathcal{O})$ |
| d) | $\mathcal{D}(\mathcal{O})$ | to | $\mathcal{D}(\mathcal{O})$. |

2) Let D^β and h be as before and let x_0 be a fixed point in \mathcal{O} . Then $T(f) = h(x_0) D^\beta f(x_0)$ and $T(f) = \int h(x) D^\beta f(x) dx$ define linear functionals on $\mathcal{E}^{[\beta]}(\mathcal{O})$ and $\mathcal{D}^{[\beta]}(\mathcal{O})$ respectively.

3) Let $H(x, y)$ be a function in $\mathcal{E}^0(\mathcal{O} \times \mathcal{O})$. Then

$$(Tf)(x) = \int H(x, y) f(y) dy$$

defines a linear function T from $D^0(\mathcal{O})$ to $\mathcal{E}^0(\mathcal{O})$. Modify H so that we get a linear function from

- a) $\mathcal{D}^0(\mathcal{O})$ to $\mathcal{E}^k(\mathcal{O})$ or $\mathcal{E}(\mathcal{O})$
- b) $\mathcal{D}^1(\mathcal{O})$ to $\mathcal{D}^k(\mathcal{O})$ or $\mathcal{D}(\mathcal{O})$
- c) $\mathcal{D}^0(\mathcal{O})$ to $\mathcal{E}^k(\mathcal{O})$ or $\mathcal{E}(\mathcal{O})$
- d) $\mathcal{E}^0(\mathcal{O})$ to $\mathcal{D}^k(\mathcal{O})$ or $\mathcal{D}(\mathcal{O})$.

How can one get antilinear functions in all these cases?

Exercise. — Two linear spaces L and L' are said to be (linearly) isomorphic if there exists a linear function T from L to all of L' such that T^{-1} is also a function (necessarily linear). Show that two finite-dimensional spaces are linearly isomorphic if and only if they have the same dimension. For infinite-dimensional spaces, the linear isomorphisms (*i.e.* the functions T above) are in practice uninteresting but the linear and bicontinuous isomorphisms are important. See the fifth section.

We have seen that if $h \in \mathcal{E}^0(\mathcal{O})$, then

$$T(f) = \int f(x) h(x) dx,$$

defines a linear functional on $\mathcal{D}^0(\mathcal{O})$. We shall later deal with more general linear functionals induced by (Radon) measures. A non-negative measure σ in \mathcal{O} is, by definition, a completely additive non-negative function defined on those Borel subsets of \mathcal{O} whose closures are compact subsets of \mathcal{O} . For every measure σ , the integral

$$(2) \quad T(f) = \int f(x) d\sigma(x),$$

is a linear functional on $\mathcal{D}^0(\mathcal{O})$ which determines the measure uniquely. It has the property that $f \geq 0 \Rightarrow T(f) \geq 0$ and

$$|T(f)| \leq \sigma(K) \max |f(x)|$$

where K is the support of f and $\sigma(K) = \int_K d\sigma$ is its measure. More generally shall consider complex measures of the form

$$d\mu = d\sigma_1 - d\sigma_2 + i(d\sigma_3 - d\sigma_4),$$

where $\sigma_1, \dots, \sigma_4$ are non-negative measures. If $T(f) = \int f d\mu$, we have

$$|T(f)| \leq c \max |f(x)|,$$

where $c = \sigma_1(K) + \dots + \sigma_4(K)$.

The simplest measure is a unit mass in a point y of \mathcal{O} . If this measure is called σ_y we have

$$\int f(x) d\sigma_y(x) = f(y).$$

Exercise. — A partition of unity in \mathcal{O} is a sequence of functions h_j in $\mathcal{D}(\mathcal{O})$ such that $h_j \geq 0$ and $1 = \sum h_j(x)$ for all x and at most a finite number of their supports intersect a given compact set. Using the fact that there exists a partition of unity in \mathcal{O} , show that to every measure μ there is a continuous function $h(x)$ such that $|\sum f d\mu| \leq \max |h(x)f(x)|$. (Hint. There are numbers $c_j > 0$ such that $|\int f h_j d\mu| \leq c_j \max |h_j f|$. Choose $e.g.$

$$h(x) = c \sum_1^\infty h_j(x)/j^2 c_j,$$

where c is a suitable constant ($6/\pi^2$.)

Exercise. — Let μ be a measure defined in all of R^n . We say that μ has at most polynomial growth if

$$\left| \int f d\mu \right| \leq c \sup | (1 + |x|)^N f(x) |,$$

for some $N > 0$, ($|x| = (\sum x_k^2)^{\frac{1}{2}}$). Verify that such a measure induces a linear functional on \mathcal{S}^0 .

Exercise. — Let μ be a measure. It is clear that there is a maximal open subset \mathcal{O}' of \mathcal{O} such that $\int f d\mu = 0$ when $f \in \mathcal{D}(\mathcal{O}')$. The complement K of \mathcal{O}' in \mathcal{O} is called the support of μ . Verify that K is the set of all points y with the following property. There are functions g in $\mathcal{D}(\mathcal{O})$ such that $\int g d\mu \neq 0$ and g vanishes outside an arbitrary open sphere containing y .

Exercise. — Let μ be a measure whose support reduces to a point y . Show that $\int f d\mu = c f(y)$, where c is a constant. (Hint. Let f be a given function with $f(y) = 0$. Verify that f is the uniform limit of functions g which vanish

in a neighborhood of y and use this to show that $f(y) = 0 \Rightarrow I(f) = \int f d\mu = 0$. Let $h(y) = 1$. Then $I(f - f(y)h) = 0$ so that $I(f) = f(y)I(h)$.

Exercise. — Show that if the support of μ consists of a sequence of points $y_j \rightarrow$ the boundary of \mathcal{O} , the $\int f d\mu = \sum c_j f(y_j)$, where the c_j are arbitrary constants.

LITERATURE

BOURBAKI: *Algèbre linéaire. Espaces vectoriels topologiques.*

SCHWARTZ: *Théorie des distributions.*

3. — Seminorms and topology (L. GÅRDING).

Let L be a linear space. A function p from L to the non-negative real numbers is said to be a seminorm if

$$\begin{aligned} (1) \quad & p(af) = |a|p(f), \\ (2) \quad & p(f+g) \leq p(f) + p(g). \end{aligned}$$

A seminorm is said to be a norm if, in addition, $p(f) = 0$ implies that $f = 0$.

Exercise. — Show that all f such that $p(f) = 0$ form a linear space M and that $p(f+g) = p(f)$ when $g \in M$ and f is any element i of L .

Exercise. — Let L have finite dimension and let f_1, \dots, f_n be a basis. Then every f in L has the form $\sum \alpha_k f_k$ where the numbers α_k are the co-ordinates of f . Let $\sigma_1, \dots, \sigma_n$ be fixed non-negative numbers. Show that $p(f) = \sum \sigma_k |\alpha_k|$ and $q(f) = (\sum \sigma_k |\alpha_k|^2)^{\frac{1}{2}}$ are seminorms which are norms if all the numbers σ_k are positive. Other examples of seminorms at the end of this section.

Exercise. — Let p be a seminorm and let U be all f in L which satisfy $p(f) \leq 1$. Verify that U contains the origin and is convex, i.e. that if $f, g \in U$ and $0 \leq \lambda \leq 1$, then $\lambda f + (1 - \lambda)g \in U$, that $aU = |a|U$, (U is symmetric), and that any f in L belongs to some λU to some λU for λ large positive (U is absorbing). Conversely, let U be any subset of L with these three properties. Verify that

$$p(f) = \inf |\lambda|,$$

where λ is such that $f \in \lambda U$, defines a seminorm p . Draw a picture of a typical U .

Exercise. — Let $a > 0$ and let p and q be seminorms. Show that ap , $p+q$ and $\max(p, q)$ are seminorms. Two seminorms are said to be equivalent if $c_1 p(f) \leq q(f) \leq c_2 p(f)$ for all f and fixed positive constants c_1 and c_2 . Show that $p+q$ and $\max(p, q)$ are equivalent and, more generally, that $p_1 + \dots + p_m$ and $\max(p_1, \dots, p_m)$ are equivalent.

A collection π of seminorms p on a linear space is said to be separating if, for every $f \neq 0$ in L , there is a $p \in \pi$ with $p(f) \neq 0$. In particular, any π which contains a norm is separating.

Exercise. — Let $\pi = \{p\}$ be a collection of seminorms on a linear space L . Show that all elements f in L which have the property that $p(f) = 0$ for all $p \in \pi$ form a linear space L_0 and that $p(f+g) = p(f)$ for any f and all $g \in L_0$. Thus p is constant on subsets of L of the form $\xi = f + L_0$. Show that if we put $p(\xi) = p(f)$, then p becomes a seminorm on the quotient space $L_1 = L/L_0$. Verify that in this way, π becomes a separating collection of seminorms on L_1 .

A separating collection π of seminorms on a linear space defines (induces) what is called a (linear, convex) topology τ on L in the following way. We define an interval around $g \in L$ to be all f such that

$$p_j(f - g) \leq \varepsilon_j, \quad (j = 1, 2, \dots, N),$$

where p_1, \dots, p_N is any finite collection of seminorms in π and $\varepsilon_1, \dots, \varepsilon_N$ are any positive numbers. Next definition: a neighborhood of $g \in L$ is any subset of L which contains an interval around g .

Remark. — Our terminology is justified by the fact that our neighbourhoods fulfil the axioms of neighborhoods in a topological space.

Exercise. — Show that if N is a neighborhood or an interval around 0, then $g + N$ is a neighborhood or an interval respectively around g . Show that the assumption that π is separating means that all the neighborhoods of any « point » g have only that point in common.

Let us now identify the topology τ with the collection of neighborhoods of the origin which we get from a separating set π of seminorms (since the other neighborhoods are simply translates of these, we can leave them out). Then we can compare topologies. We say that $\tau_1 \subset \tau_2$ if every τ_1 -neighborhood is also a τ_2 -neighborhood. Then, if $\tau_1 \subset \tau_2$ and $\tau_1 \supset \tau_2$, the two topologies are the same.

Exercise. — Show that $\tau_1 \subset \tau_2$ if and only if every τ_1 -neighborhood contains a τ_2 -neighborhood.

Exercise. — Let π_1 and π_2 induce the topologies τ_1 and τ_2 . Show that $\tau_1 \subset \tau_2$ if and only if π_2 dominates π_1 in the sense that to every $p \in \pi_1$ there are seminorms $q_1, \dots, q_m \in \pi_2$ and a constant $c > 0$ such that $p \leq c \max(q_1, \dots, q_m)$. Corollary: $\tau_1 = \tau_2$ if and only if π_1 and π_2 are equivalent in the sense that π_1 dominates π_2 and conversely. Two norms induce the same topology if and only if they are equivalent.

Exercise. — Show that any two norms on a finite-dimensional space are equivalent. Show also that any two separating collections of seminorms on a finite-dimensional space induce the same topology. (Hint. Let f_1, \dots, f_n be a basis and choose to every f_j a seminorm $p_j \in \pi$ such that $p_j(f_j) > 0$. Show that π induces the same topology as $p = \max(p_1, \dots, p_n)$.)

Exercise. — Let L be the linear space of all bounded complex-valued functions on the integers. Show that the two norms

$$p(f) = \sum p_k |f(k)|, \quad q(f) = \sum q_k |f(k)|,$$

where all p_k and q_k are positive, and $\sum p_k$ and $\sum q_k$ converge, are equivalent if and only if the numbers q_k/p_k and p_k/q_k are bounded. This shows that L has different topologies.

Let us now define the usual topologies of the spaces $\mathcal{E}^k(\mathcal{O})$, $\mathcal{E}(\mathcal{O})$, $\mathcal{D}^k(\mathcal{O})$, and $\mathcal{D}(\mathcal{O})$. They can all be defined by seminorms of the type

$$p_\varphi(f) = \sup_x \sum_\alpha |\varphi_\alpha(x)| |D_\alpha f(x)|,$$

where $\varphi = \{\varphi_\alpha\}$ is a set of continuous functions subject to the following restrictions.

$\mathcal{E}^k(\mathcal{O})$: every φ_α has compact support and vanishes when $|\alpha| > k$.

$\mathcal{E}(\mathcal{O})$: every φ_α has compact support and vanishes when $|\alpha|$ is large enough.

$\mathcal{D}^k(\mathcal{O})$: no restrictions on the growth of φ_α at the boundary of \mathcal{O} , but $\varphi_\alpha = 0$ when $|\alpha| > k$.

$\mathcal{D}(\mathcal{O})$: no restrictions on the growth of φ_α at the boundary of \mathcal{O} , but to every compact subset K of \mathcal{O} there is an integer m (depending on K and the collection φ) such that $\varphi_\alpha = 0$ on K when $|\alpha| > m$.

Exercise. — Let $K_1 \subset K_2 \subset \dots$ be a sequence of compact sets which exhaust \mathcal{O} . Show that we get equivalent sets of seminorms in all cases if we replace the condition that φ_α be continuous by the condition that φ_α is continuous on all $K_{j+1} - K_j$.

The topology \mathcal{S}^m is defined by the following simple set of seminorms,

$$p_{k,j}(f) = \sup |x_\alpha D^\beta f(x)|, \quad |\alpha| \leq k, \quad |\beta| \leq j,$$

where $k = 0, 1, 2, \dots$ and $j \leq m$. For the space \mathcal{S} we take the same seminorms with j unrestricted. Let L be a linear space with a topology τ induced by a set of seminorms $\pi = \{p\}$ and let M be a subset of L . We say that f is a limit point (element) of M if any neighborhood of f contains elements in M . In our case, an equivalent statement is the following. Given any finite set

of seminorms p_1, \dots, p_m in π and positive numbers $\varepsilon_1, \dots, \varepsilon_m$, we can find a g in M such that $p_j(f - g) < \varepsilon_j$, $j = 1, \dots, m$. The set M together with this limit point constitutes the closure \overline{M} of M (relative to the topology τ). When $\overline{M} = L$, we say that M is dense in L .

Exercise. — Show that $\overline{\overline{M}} = \overline{M}$.

Exercise. — Show that $\mathcal{D}^k(\mathcal{O})$ and $\mathcal{D}(\mathcal{O})$ are dense in $\mathcal{E}^k(\mathcal{O})$ and $\mathcal{E}(\mathcal{O})$ respectively.

Exercise. — Let $\{L_j\}_1^m$, ($m = \infty$ permitted) be linear topological spaces. Their direct sum

$$L = L_1 + L_2 + \dots$$

consists of all sequences

$$f = (f_1, f_2, \dots), \quad f_j \in L_j,$$

with the natural linear operations. Let L_j have the topology τ_j induced by a set π_j of seminorms p_j . Verify that all

$$p(f) = p_1(f_1) + \dots \quad (\text{finite sum})$$

defines a separating set of seminorms on L . This defines the topology on a direct sum.

LITERATURE

BOURBAKI: *Espaces vectoriels topologiques*.

4. — Complete spaces (L. GÅRDING).

Let L be a linear space with a topology induced by a set of seminorms $\{p\}$. We say that a sequence $\{f_j\}_1^\infty$ of elements in L tends to a limit $f \in L$ if $p(f_j - f)$ tends to zero with $1/j$ for every p . This means that for every p

$$p(f_j - f_k) \rightarrow 0 \quad \text{as} \quad \min(j, k) \rightarrow \infty.$$

A sequence with this property is said to be a Cauchy sequence. The space L is said to be complete if every Cauchy sequence has a limit. (This is not quite true, one has to require the same for generalized sequences, Cauchy filters, but we do not go into that. If the topology is induced by some countable set of seminorms, it suffices to consider Cauchy sequences.)

Example. — Let L be all continuous functions from a closed bounded interval I on the real axis to the complex numbers with the single norm

$$p(f) = \sup |f(x)|, \quad x \in I.$$

If f_j is a Cauchy sequence, then $f_j(x)$ is also a Cauchy sequence for every x and consequently $f(x) = \lim f_j(x)$ exists. Also, $p(f_j - f) \rightarrow 0$ as $j \rightarrow \infty$ and it is a classical fact that f is continuous. Thus, L is complete. It is a simple example of a Banach space, i.e. a complete linear space with a single norm. Let L_0 be the set of all polynomials in L with the norm above. Then L_0 is a linear space but it is not complete. In fact, by Weierstrass' theorem any $f \in L$ can be approximated by polynomials g_j in such a way that $p(f_j - g_j) \rightarrow 0$ as $j \rightarrow \infty$.

Exercise. — Use the example to prove that the spaces $\mathcal{E}^k(\mathcal{O})$, $\mathcal{E}(\mathcal{O})$, $\mathcal{S}^k = \mathcal{S}^k(\mathcal{R}^n)$, and $\mathcal{S}(\mathcal{R}^n)$ are complete.

Exercise. — Show that if $f_j \rightarrow 0$ in $\mathcal{D}(\mathcal{O})$ then the supports of the f_j are contained in a fixed compact set. (Hint. If $f_j(x_j) \neq 0$ and $x_j \rightarrow$ the boundary of \mathcal{O} we can construct a continuous function φ such that $|\varphi(x_j)f_j(x_j)| \rightarrow \infty$. This means that $p_{\varphi}(f_j) \rightarrow \infty$ and we have a contradiction.) Use this to show that the spaces $\mathcal{D}^k(\mathcal{O})$ and $\mathcal{D}(\mathcal{O})$ are complete. Verify the statement that $f_j \rightarrow 0$ in $\mathcal{D}(\mathcal{O})$ if and only if the supports of the functions f_j are contained in a fixed compact set K and all the derivatives $D_{\alpha}f_j$ tend to zero uniformly on K . Find analogous statements for the other spaces.

A complete space with a topology which can be defined by a countable set of seminorms is called a Fréchet space. The spaces $\mathcal{E}^k(\mathcal{O})$, $\mathcal{E}(\mathcal{O})$, \mathcal{S}^k , and \mathcal{S} are Fréchet spaces. Let G be an open subset of \mathcal{O} with compact closure and let $\mathcal{D}_0^k(G)$ be all elements of $\mathcal{D}^k(\mathcal{O})$ which vanish outside G and analogously for $\mathcal{D}_0(G)$. These spaces are also Fréchet spaces, but the spaces $\mathcal{D}^k(\mathcal{O})$ and $\mathcal{D}(\mathcal{O})$ are not.

Any incomplete linear topological space L can be imbedded into a complete space \hat{L} , which we get from L by a construction involving Cauchy sequences, analogous to Cantor's construction of the real numbers from the rational ones. The space L is dense in its completion \hat{L} , which is essentially unique (in the sense of a linear topological isomorphism, see the next section). Sometimes, this universal procedure, which we do not go into, can be replaced by integration theory. We shall consider a simple case. Let σ be a non-negative measure on \mathcal{O} and let L be the set of functions in $\mathcal{E}^0(\mathcal{O})$ for which

$$p(f) = p_{\sigma}(f) = \left[\int |f(x)|^{\gamma} d\sigma(x) \right]^{1/\gamma} < \infty,$$

where $\gamma \geq 1$ is a fixed number. The right side is a seminorm. In fact, it is trivial that $p(af) = |a|p(f)$ and, by Minkowski's inequality, we have $p(f+g) \leq p(f) + p(g)$. Let L_0 be the linear space of all f for which $p(f) = 0$. When p is a norm, $L_0 = 0$. The space $L_1 = L/L_0$ is not complete in general. Its com-

pletion \widehat{L}_1 can be described as follows. Taking successive monotone limits of the real and imaginary parts of the functions in $\mathcal{E}^0(G)$ we arrive at the class B of complex Baire (Borel measurable) functions for which the integral $p(f)$ is well defined and may be $+\infty$. But since there is no unique way of adding or multiplying complex numbers of the form $\alpha + i\beta$, where α or α or both are $+\infty$ or $-\infty$, B fails to be a linear space. In fact, when $f, g \in B$, af and $f+g$ are not defined in certain points where the values of the functions are infinite.

Let us, however, restrict ourselves to the set B_p of all Baire functions with $p(f) < \infty$. Then one can prove that the set E of \mathcal{O} where $f \in B_p$ is not finite is a set of σ -measure zero in the sense that if $B_p \ni g = 0$ outside E then $p(g) = 0$. This means that if we modify the definition of B_p so that it consists of all Baire functions f with $p(f) < \infty$ which are defined *except in sets of τ -measure zero* (depending on f), then B_p becomes a linear space and the set B_p^0 of functions f in B_p for which $p(f) = 0$ form a linear subspace. The completion of L_1 can now be described as the space B_p/B_p^0 , equipped with the norm p . Space of this kind with $\gamma = 2$ play a great role in the theory of Hilbert spaces. Their complicated nature makes them almost inpalatable to a physicist, but in mathematics they are as inevitable as the real numbers.

The theorem that B_p/B_p^0 is complete was first proved by F. RIESZ and FISHER (1908) in the special case when \mathcal{O} is an interval on the real axis, $\gamma = 2$ and $d\sigma = dx$.

Exercise. — Let σ be concentrated to a sequence of points, x_1, x_2, \dots , and put $\sigma_k = \sigma(x_k)$. If the sequence has a limit just inside \mathcal{O} , suppose that $\sum \sigma_k < \infty$. Then

$$p(f) = [\sum |f(x_k)|^\gamma \sigma_k]^{1/\gamma}.$$

Show that a subset of \mathcal{O} is a nullset if and only if it does not contain any point x_k . If σ is an ordinary Lebesgue measure on \mathcal{O} , the nullsets are those of Lebesgue measure zero.

LITERATURE

BOURBAKI: *Intégration*.

5. — Continuous linear functions (L. GÅRDING).

Let L and L' be two linear topological spaces with topologies τ and τ' respectively. A function (transformation) T from L to L' is said to be continuous at $f \in L$ if, to every neighborhood N' of Tf there is a neighborhood N of f such that $T(N) \subset N'$. For linear functions defined in all of L we have the following theorem.

Theorem 5'1. – The following three conditions on T are equivalent:

1) T is continuous.

2) T is continuous at the origin.

3) To every seminorm p' on L there are seminorms p_1, \dots, p_m on L and a positive number c such that

$$(1) \quad p'(Tf) \leq c(p_1(f) + \dots + p_m(f))$$

for all f in L .

Proof. – We shall prove the logical chain $1) \Rightarrow 2) \Rightarrow 3) \Rightarrow 1)$. Its first link is obvious. Consider the second. Since $p'(f') \leq 1$ defines a neighborhood of the origin in L' we know from the continuity of T at the origin that there exist seminorms p_1, \dots, p_m and a number $\varepsilon > 0$ such that

$$(2) \quad p(g) \leq \varepsilon \Rightarrow p'(Tg) \leq 1,$$

where $p = p_1 + \dots + p_m$. The left inequality holds for $g = \varepsilon f / p(f)$ where f is any element with $p(f) \neq 0$. Hence

$$p' \left(T \frac{\varepsilon f}{p(f)} \right) \leq 1,$$

i.e.

$$(3) \quad p'(Tf) \leq \varepsilon^{-1} p(f),$$

whenever the right side does not vanish. If it vanishes, we get from (2) that $1 \geq p'(Taf) = |a| p'(Tf)$ for any complex number a and consequently $p'(Tf) = 0$. Hence (3) holds for every f . This proves that $2) \Rightarrow 3)$. The proof that $3) \Rightarrow 1)$ is left as an exercise for the reader.

Remark. – When L and L' are normed spaces with norms p and p' respectively then (1) reads

$$p'(Tf) \leq c p(f).$$

In view of this, continuous linear transformations are often characterized as being *bounded*. We say that $B \subset L$ is bounded if $p(B)$ is bounded for every seminorm p . Verify that a linear continuous function sends bounded sets into bounded sets. When the topology of L can be defined by a countable number of seminorms, the converse of this proposition is also true.

Exercise. — Let L have finite dimension. Show that any linear function from L to L' is continuous. (Let f_1, \dots, f_m be a basis of L . Write $f = \sum a_k f_k$. Then $p'(Tf) \leq \sum a_k p'(Tf_k)$ and the right side is a seminorm on L .)

Exercise. — Verify that all the linear transformations T defined at the end of Section 1 are continuous, when the spaces have their usual topologies. (Hint. Take e.g. $(Tf)(x) = h D_\beta f(x)$ considered as a transformation from $\mathcal{E}^{k+|\beta|}(\mathcal{O})$ to $\mathcal{E}^k(\mathcal{O})$. We have

$$p_\varphi(Tf) = \sup \sum |h(x) \varphi_\alpha(x) D_\alpha D_\beta f(x)|, \quad |\alpha| \leq k,$$

where the right side, by definition, is a seminorm in $\mathcal{E}^{k+|\beta|}(\mathcal{O})$.)

Exercise. — Let $h \in \mathcal{E}(R^n)$ be of a most polynomial growth and let D_β be a fixed derivative. Show that $(Tf)(x) = h(x) D_\beta f(x)$ defines a continuous linear mapping from \mathcal{S} to \mathcal{S} .

Exercise. — Let L be $\mathcal{D}(\mathcal{O})$ with the topology defined by the single norm $p(f) = \max |f(x)|$. Let D_β , ($|\beta| > 0$), be a derivative. Show that $(Tf)(x) = D_\beta f(x)$ defines a discontinuous linear transformation.

(This is no contradiction. We have taken a different topology on $\mathcal{D}(\mathcal{O})$. When nothing else is said, we mean by $\mathcal{D}(\mathcal{O})$, $\mathcal{D}^k(\mathcal{O})$, $\mathcal{E}(\mathcal{O})$, $\mathcal{E}^k(\mathcal{O})$ and \mathcal{S} these spaces equipped with the standard topologies.)

Continuous linear functionals on $\mathcal{D}(\mathcal{O})$ are called *distributions*. We shall get explicit expressions for them in this section and analyse them more closely in the next section.

Let L be an arbitrary linear topological space equipped with a topology τ induced by a set of seminorms $\pi = \{p\}$. All continuous linear functionals in L form a linear space L^* ; called the dual space of L . The following celebrated theorem tells us that L^* has plenty of elements. For convenience we shall write linear functional instead of continuous linear functional from now on.

THEOREM 5'2 (Hahn-Banach). — Let L be a linear topological space. Any linear functional A defined in a subspace M of L can be continued to a linear functional A_1 defined on the whole of L . If $|A(f)| \leq p(f)$, ($f \in M$), for some seminorm p , then A_1 can be chosen so that $|A_1(f)| \leq p(f)$ for all f in L .

We shall not prove this theorem, but we are going to apply it.

Exercise. — Verify that there exists an $A \in L^*$ which reduces to a given linear functional on a given finite-dimensional subspace M . Use this to prove that if $f_0 \in L$ and $A(f_0) = 0$ for all A in L^* then $f_0 = 0$. (Hint. Construct a suitable linear functional on the space spanned by f_0 .)

Exercise. — Let $L = L_1 + L_2 + \dots$ be a direct sum of linear spaces L_j . Show that every element T of L^* has the form

$$T(f) = \sum T_j(f_j),$$

where $T_j \in L_j^*$ and the sum is finite.

We can now describe the duals of the spaces $\mathcal{D}^k(\mathcal{O})$, $\mathcal{D}(\mathcal{O})$, $\mathcal{E}^k(\mathcal{O})$, $\mathcal{E}(\mathcal{O})$ and \mathcal{S} .

THEOREM 5'3. — Every element T of $\mathcal{D}^0(\mathcal{O})^*$ has the form

$$T(f) = \int f \, d\mu,$$

where μ is a complex measure.

We do now prove this theorem which is essentially due to F. RIESZ (1909).

THEOREM 5'4. — Every element T of $\mathcal{D}^k(\mathcal{O})^*$ has the form

$$T(f) = \sum \int D_\alpha f \, d\mu_\alpha, \quad |\alpha| \leq k,$$

where the μ_α are arbitrary measures.

Exercise. — Let L_α , $|\alpha| \leq k$, be copies of the space $\mathcal{D}^0(\mathcal{E})$ and let $L = \sum L_\alpha$ be their direct sum. Using the linear mapping

$$\mathcal{D}^k(\mathcal{O}) \ni f \rightarrow \{D_\alpha f\}, \quad |\alpha| \leq k,$$

we can identify $\mathcal{D}^k(\mathcal{O})$ with a linear subspace of L . Use the Hahn-Banach theorem to prove Theorem 5'4.

THEOREM 5'5. — Every element T of $\mathcal{D}(\mathcal{O})^*$ has the form

$$T(f) = \sum \int D_\alpha f \, d\mu_\alpha,$$

where the μ_α are measures with the property that to any compact subset K of \mathcal{O} , there is an N such that $\mu_\alpha = 0$ on K when $|\alpha| \geq N$.

Exercise. — Let \mathcal{O}' be an open set with compact closure in \mathcal{O} . Show that every $T \in \mathcal{D}(\mathcal{O})^*$ when restricted to $\mathcal{D}^k(\mathcal{O}')$ gives an element of some $\mathcal{D}^k(\mathcal{O}')$ and use this observation to prove Theorem 5'5. (Use a partition of unity.)

THEOREM 5'6. — Every element T of $\mathcal{E}^0(\mathcal{O})^*$ has the form

$$T(f) = \int f d\mu,$$

where μ has compact support in \mathcal{O} .

Exercise. — The notion of support, previously given for measures, carries over to any T in the duals of the spaces we are dealing with now. Show that any $T \in \mathcal{E}^0(\mathcal{O})^*$ has compact support and use this to prove Theorem 5'6. (Hint. If T did not have compact support, we can find a sequence h_j of functions in $\mathcal{D}(\mathcal{O})$ whose supports tend to the boundary of \mathcal{O} and are such that $T(\varphi_j) = 1$. Verify that $\varphi_j \rightarrow 0$ in $\mathcal{E}^0(\mathcal{O})$. This gives a contradiction.)

THEOREM 5'7. — Every element T of $\mathcal{E}^k(\mathcal{O})^*$ has the form

$$T(f) = \sum \int D_\alpha f d\mu_\alpha, \quad |\alpha| \leq k,$$

where the μ_α have compact supports.

THEOREM 5'8. — Every element T of $\mathcal{E}(\mathcal{O})^*$ has the form

$$T(f) = \sum \int D_\alpha f d\mu_\alpha,$$

where the μ_α have compact supports and vanish for large enough $|\alpha|$.

Exercise. — Prove these theorems using the method of proof of Theorem 5'4.

THEOREM 5'9. — Every T in \mathcal{S}^{j*} or \mathcal{S}^* has the form

$$T(f) = \sum \int D_\alpha f d\mu_\alpha,$$

where the μ_α are of at most polynomial growth and $\mu_\alpha = 0$ when $|\alpha| \geq j$ or when $|\alpha|$ is large enough, respectively.

Exercise. — Prove Theorem 5'9 by reducing it to the special case $j = 0$.

Exercise. — Let L be a linear topological space and let L^* be its dual. Verify that

$$(1) \quad p_f(T) = |T(f)|; \quad f \text{ fixed in } L,$$

is a seminorm on L^* . We recall that a subset B of L is said to be bounded

if, for every seminorm p , the set $p(B) = \{p(g) \mid g \in B\}$, is bounded. Verify that,

$$(2) \quad p_B(T) = \sup \{ |T(f)| \mid f \in B \},$$

is a seminorm on L^* . Verify that (1) and (2) define separating sets of seminorms π and π' on L^* and that $\pi \subset \pi'$. These sets induce topologies on L^* , called the weak and strong topologies respectively. Accordingly, one speaks of L^* as being the weak and strong dual of L . We shall use the strong dual. It is convenient to use the notation

$$F(f) = \langle F, f \rangle,$$

where $f \in L^*$ and $f \in L$.

Exercise. — Verify that for fixed $f \in L$, $\langle F, f \rangle$ is a linear continuous function of $F \in L^*$.

Let L and M be linear spaces and let A be a continuous linear function from L to M . Then $\langle F, Af \rangle$, ($F \in M^*$, $f \in L$) is a linear continuous function of $f \in L$ and consequently there is an $A^*F \in L^*$ such that

$$\langle A^*F, f \rangle = \langle F, Af \rangle, \quad (\text{all } f \text{ in } L).$$

Exercise. — Verify that A^* is a continuous linear mapping from M^* to L^* . It is called the adjoint of A .

Exercise. — Let $Af = \sum \langle F_k, f \rangle g_k$, where F_k and g_k are fixed elements in L^* and M respectively. Show that

$$A^*G = \sum \langle G, g_k \rangle F_k,$$

where $G \in M^*$.

We have the following simple and important theorem, which we shall use frequently in the special case when $M = L$.

THEOREM 5'10. — If A^{-1} is continuous, then $A^*M^* = L^*$.

Proof. — Let $F \in L^*$, then $\langle F, f \rangle$ is a continuous function of $g = Af$. Extending it from AL to M , we see that $G \in M^*$ exists such that $\langle F, f \rangle = \langle G, Af \rangle$ for all $f \in L$, i.e. $A^*G = F$.

Remark. — If L is a Fréchet space, the converse of this theorem is true.

Exercise. — Use the theory of linear systems of equations to prove Theorem 5'10 when L and M are finite-dimensional.

Let L be a linear topological space and let L^* be its adjoints. We have seen that

$$\langle F, f \rangle, \quad (F \in L^*, f \in L),$$

is a continuous linear function of F when f is fixed. If every linear functional on L^* has this form, we say that L is semi-reflexive. If, in addition, the topologies on L and L^{**} coincide, we say that L is reflexive. It can be shown that the spaces $\mathcal{E}(\mathcal{O})$, $\mathcal{D}(\mathcal{O})$ and $\mathcal{S}(R^n)$ are reflexive.

Exercise. — Show that every finite-dimensional space is reflexive. Show that the space of summable sequences is not reflexive. (Its dual are the bounded sequences and has more continuous linear functionals than those given by the summable sequences.)

LITERATURE

BOURBAKI: *Espaces vectoriels topologiques*.

SCHWARTZ: *Théorie des distributions*.

6 - The technical theory of distributions (L. GÄRDING).

6.1. *Introduction.* — We have defined distributions as continuous linear functionals on $\mathcal{D} = \mathcal{D}(\mathcal{O})$ and we know that every distribution has the form given in Theorem 5.5. In particular, every function g in $\mathcal{E}^0(\mathcal{O})$ gives rise to the distribution

$$\langle g, h \rangle = \int g(x) h(x) dx,$$

which we shall identify with g . It is useful to extend the notation on the left side by writing

$$\langle T, h \rangle = T(h)$$

for an arbitrary distribution T . Sometimes we shall also abuse the notation of the integral, writing

$$(6.1) \quad \langle T, h \rangle = \int T(x) h(x) dx,$$

although it does not make sense to speak of T as a function.

Example. — When \mathcal{O} contains the origin, $\langle \delta, h \rangle = h(0)$ defines a distribution δ , Dirac's distribution. Improperly,

$$h(0) = \int h(x) \delta(x) dx,$$

where δ is Dirac's « function ». Sometimes one writes $\delta(x_1) \dots \delta(x_n)$ instead of $\delta(x)$.

6.2. Supports. — Let T be a distribution and let \mathcal{O}' be the largest open subset of \mathcal{O} such that T vanishes in \mathcal{O}' in the sense that

$$h \in \mathcal{D}(\mathcal{O}') \Rightarrow \langle T, h \rangle = 0.$$

The complement of \mathcal{O}' in \mathcal{O} is called the support of T and we shall denote it by $\text{supp } T$.

Example. — $\text{Supp } \delta$ is the origin, and $\text{supp } x_1 \dots x_n$ is the whole space.

Exercise. — Let $K = \text{supp } T$ be compact. Verify that T extends in a natural way from $\mathcal{D}(\mathcal{O})$ to $\mathcal{E}(\mathcal{O})$. (Let $\varphi \in \mathcal{D}(\mathcal{O})$ be 1 in a neighborhood of K . Define $\langle T, h \rangle = \langle T, \varphi h \rangle$ when $h \in \mathcal{E}(\mathcal{O})$). Verify that the space $\mathcal{E}(\mathcal{O})^*$ is identical with the set of those elements of $\mathcal{D}(\mathcal{O})^*$ which have compact supports.

6.3. Order. — Let $\mathcal{O}' \subset \mathcal{O}$. The order of T in \mathcal{O}' is defined as the least integer m for which

$$|\langle T, h \rangle| \leq \max |D^\alpha h(x)|, \quad |\alpha| \leq m; \quad h \in \mathcal{D}(\mathcal{O}').$$

When no such integer exists, we say that the order of T is infinite.

Exercise. — Let $\mathcal{O} = \mathcal{O}' = R$ and put $\langle T, h \rangle = \sum_{j=-\infty}^{+\infty} h^{(j)}(j)$. Verify that the order of T is infinite. Let $\mathcal{O}' \subset \mathcal{O}$ be compact. Verify that the order of T is finite in \mathcal{O}' .

When K is compact, the order of T is defined as the infimum of the orders of T in the open sets \mathcal{O}' such that $\mathcal{O}' \supset K$.

A derivative $D^\alpha T$ of a distribution T is defined by the formula

$$\langle D^\alpha T, h \rangle = (-1)^{|\alpha|} \langle T, D^\alpha h \rangle.$$

Verify that this coincides with the ordinary definition when $T \in \mathcal{E}(\mathcal{O})$.

Example. — One has

$$\langle D^\alpha \delta, h \rangle = (-1)^{|\alpha|} D^\alpha h(0).$$

Exercise. — A measure μ gives rise to the distribution defined by $\int h d\mu$. We shall also denote it by $\langle \mu, h \rangle$. Verify that every distribution T is a sum $\sum c_\alpha D^\alpha \mu_\alpha$ and specify the sum and the supports of the measures according as $T \in \mathcal{D}(\mathcal{O})^*$, $\mathcal{D}^k(\mathcal{O})^*$ and so on.

Exercise. — Verify that $T \rightarrow D^\alpha T$ is a continuous mapping. (It follows from the fact that $h \rightarrow (-1)^{|\alpha|} D^\alpha h$, ($h \in \mathcal{D}(\mathcal{O})$), is continuous.)

6.4. Distributions whose supports are a point.

THEOREM 6'1. - Any distribution T whose support is the origin has the form $\sum c_\alpha D^\alpha \delta$, (finite sum) and conversely.

Exercise. - Prove the theorem by verifying the following statements.

- 1) $T \in \mathcal{E}^m(R^n)^*$ for some $m > 0$.
- 2) Let $\varphi_1 \in \mathcal{E}$ be 1 for $|x| \geq 1$ and 0 for $|x| \leq \frac{1}{2}$. Then $|x|^{2m} \varphi_1(x/\varepsilon)$, ($\varepsilon \rightarrow 0$), tends to $|x|^{2m}$ in $\mathcal{E}^m(R^n)$.
- 3) $\langle T, h \rangle = 0$ when h vanishes of order $2m$ at the origin.
- 4) Let Ph be the beginning of a power series for h at the origin to order $2m$. Then $\langle T, h \rangle = \langle T, Ph \rangle$.

Exercise. - Formulate the theorem for an arbitrary point instead of the origin.

6'5. *Regularization.* - Let $\varphi_1 \in \mathcal{D}$ have the property that $\varphi_1 \geq 0$ and $\int \varphi_1(x) dx = 1$. Then, if $\varphi(x) = \varepsilon^{-n} \varphi_1(x/\varepsilon)$, one has

$$\langle \delta, h \rangle = \lim_{\varepsilon \rightarrow 0} \langle \varphi, h \rangle,$$

so that δ is a limit of functions.

The same process can be applied to an arbitrary distribution if we introduce a convolution $T * \varphi$ defined by

$$(T * \varphi)(x) = \int T(y) \varphi(x - y) dy.$$

Then $T * \varphi \in \mathcal{E}(\mathcal{O}')$ if $\mathcal{O}' + \varepsilon K \subset \mathcal{O}'$, ($K = \text{supp } \varphi_1$), and

$$\langle T * \varphi, h \rangle = \langle T, h * \check{\varphi} \rangle, \quad (\check{\varphi}(x) = \varphi(-x)).$$

Since $h * \varphi \rightarrow h$ in \mathcal{D} as $\varepsilon \rightarrow 0$, we get that

$$\langle T * \varphi, h \rangle \rightarrow \langle T, h \rangle, \quad (\varepsilon \rightarrow 0).$$

The function $T * \varphi$ is called a regularization of T .

Exercise. - Show that if $\mathcal{O} = R^n$ then $\text{supp } T * \varphi \subset \text{supp } T + \varepsilon K$.

Exercise. - Let $n = 1$ and let $\varphi_\varepsilon \in \mathcal{D}$ approximate δ . Show that $\varphi_\varepsilon^{(k)}$ approximates $\delta^{(k)}$.

6.6. *Solutions of differential equations.* — Let $a(\zeta_1, \dots, \zeta_n)$ be a polynomial and consider the differential equation

$$(1) \quad a(D)F = G,$$

where F and G are distributions. We shall call F a weak solution of the equation. A weak solution is said to be a strong solution if it is a function and satisfies the equation in the ordinary sense. (This implies that G is also a function.)

Exercise. — Show that if $a(D)F = 0$, then $\varphi * F$ is an ordinary solution of the same equation. (Show first that $a(\varphi * h) = \varphi * ah$, when $h \in \mathcal{D}$.) Thus every solution of the homogeneous equation is a limit of ordinary solutions.

LEMMA 6.1. — Let $n = 1$ and $G = 0$. Show that every solution of (1) is ordinary.

Exercise. — Prove the Lemma as follows. Let F_1, \dots, F_m be a basis for the ordinary solutions (they are of the form $p(x)e^{\lambda x}$) and choose $h_1, \dots, h_m \in \varepsilon$ such that $\langle F_j, h_k \rangle = \delta_{jk}$. Develop $\varphi * F = \sum c_k F_k$. Show that the numbers c_k converge as $\varepsilon \rightarrow 0$.

Exercise. — Show that the mapping $\mathcal{D}(\mathcal{O}) \ni h \rightarrow Dh \in \mathcal{D}(\mathcal{O})$ has a continuous inverse and deduce from this that $h \rightarrow a(D)h$ also has a continuous inverse. Use this to show that (1) has a solution F for every $G \in \mathcal{D}(\mathcal{O})^*$. This result is true in any number of variables. In particular, $a(D)F = \delta$ has at least one solution F . Verify that $a(F * f) = f$, ($f \in \mathcal{D}$). The distribution $F(x - z)$ is called an elementary solution of a . Verify that $F_z(x) = c(\sum x_k - z_k)^2)^{(2-n)/2}$ ($n > 2$, c a suitable constant), is an elementary solution (in this case a function) for $a(D) = D_1^2 + \dots + D_n^2$.

Exercise. — Let $n = 1$ and $a = D^m + \text{lower terms}$. Show that $a(D)F = \delta$ if and only if $a(D)F = 0$ outside the origin and $F^{(j)}(+0) - F^{(j)}(-0) = (-1)^m \delta_{j, m-1}$, ($j \leq m-1$). Show that $G(x, y) = F(x - y)$ is a Green's function for $a(D)$.

Lemma 1 fails to hold for more than one variable. When $a(D) = \sum a_\alpha D^\alpha$, $\alpha_1 \leq m$, is of order m , we define its principal part $\mathcal{P}a$ by $\mathcal{P}a(D) = \sum a_\alpha D^\alpha$, $\alpha_1 = m$. We say that a is elliptic if the polynomial $\mathcal{P}a(\zeta)$ never vanishes for ζ real and $\neq 0$. We have

LEMMA 6.2. — If a is elliptic, then every solution of (1) with $G = 0$ is a function in $\mathcal{E}(\mathcal{O})$.

Remark. — Actually, the solutions are analytic functions. The largest class of operators for which the lemma holds are characterized by the following

property: $a(i\zeta + \eta) \rightarrow \infty$ as $\zeta \rightarrow \infty$, uniformly when η is bounded. We shall not prove Lemma 6'2.

Exercise. — Let $T \in \mathcal{D}(R^2)^*$ and $T_1, T_2 \in \mathcal{D}(R)^*$. Interpret the equation $T(x_1, x_2) = T_1(x_1) + T_2(x_2)$ and show that $D_1 D_2 T = 0$. Hence Lemma 2 fails to hold for $a = D_1 D_2$.

Let $x = (x', x'')$ be a division of the co-ordinates x into two sets with m and $n - m$ elements respectively. Let $\varphi \in \mathcal{D}(R^{n-m})$ and $T \in \mathcal{D}(R^n)^*$. The convolution $\varphi * T$ is defined in the usual way,

$$\langle \varphi * T, h \rangle = \langle T, \check{\varphi} * h \rangle,$$

where $(\check{\varphi} * h)(x) = \int \varphi(y'' - x'') h(x', y'') dy''$. We have a *partial* regularization of T and, in general, $\varphi * T$ is a distribution. When T is a solution of a differential equation $a(D)T = 0$, it may happen that $\varphi * T$ is infinitely differentiable. This happens *e.g.* when

$$a(D) = \sum a_\alpha(D') D''^\alpha$$

and $a_0(D')$ is elliptic and degree $a_0 > \text{degree } a_\alpha$ when $|\alpha| > 0$. Taking $m = n$, we get Lemma 6'2 as a special case. (GÄRDING and MALGRANGE).

Exercise. — Let $x' = x_1$ and assume that $a(D) = cD_1^l + \text{lower terms in } D_1$, ($c \neq 0$). Verify that if $a(D)T = 0$, then $\varphi * T$, ($\varphi = \varphi(x'')$) is infinitely differentiable. Find a corresponding statement for the wave operator $D_1^2 - D_2^2 \dots$ (take $x'' = x_1$).

With a few reservations the results of this section hold also for differential operators with coefficients in $\mathcal{E}(\mathcal{O})$.

6'7. *Distributions independent of one or more variables.* — Let $x = (x', x'')$ be a division of variables. We say that $F \in \mathcal{D}(\mathcal{O})^*$ is independent of x'' if $D_k T = 0$ for all $x_k \in x''$. If \mathcal{O} is connected, such an F can be described as follows. Let $(Nh)(x') = \int h(x', x'') dx''$. Then $\langle F, h \rangle = \langle G, Nh \rangle$, where $G \in \mathcal{D}(\mathcal{O}')^*$ and $\mathcal{O}' = \text{the range of } x' \text{ as } x \in \mathcal{O}$. Conversely, every distribution $\langle G, Nh \rangle$ is independent of x'' . We shall not prove this statement. It could be stated briefly and intuitively as $F(x', x'') = G(x')$.

6'8. *Multiplication and division.* — Let $f \in \mathcal{D}(\mathcal{O})$ and $T \in \mathcal{D}(\mathcal{O})^*$. We define a product fT by

$$\langle fT, h \rangle = \langle T, fh \rangle.$$

Exercise. — Verify that the mapping $T \rightarrow fT$ is continuous.

The problem of division is to solve the equation $fT = S$ for T . Let us put $n = 1$, $f = x$ and $S = 1$. Then $F = x^{-1}$ outside the origin and one solu-

tion is $F = Px^{-1}$, where P denotes principal value so that

$$\langle Px^{-1}, h \rangle = \lim_{\varepsilon \rightarrow 0} \int_{|x| \geq \varepsilon} x^{-1} h(x) dx.$$

Exercise. — Verify that $Px^{-1} \in \mathcal{D}(R)^*$ and that the general solution of $xF = 1$ is $F = Px^{-1} + c\delta$.

Exercise. — Show that $\mathcal{D} \ni h \rightarrow xh \in \mathcal{D}$ has a continuous inverse. This means that $x\mathcal{D}^* = \mathcal{D}^*$. Get the same result when x is replaced by a polynomial. (The mean value theorem shows that $\max |D^k h(x)| \leq c_k \max |D^{k+1} xh(x)|$).

Remark. — The same result holds for several variables and if \mathcal{D} is replaced by \mathcal{S} (LOJASIEWICZ, HÖRMANDER).

Exercise. — Solve the equation $xF = \delta$ (one variable).

6.9. Distribution with supports in a closed set. — Let C be the closure of an open set in R^n and let $\mathcal{D}(C)$ be the space of the restricts $\mathcal{D}g$ to C of all $g \in \mathcal{D}$. The topology of $\mathcal{D}(C)$ is defined by the seminorms used in \mathcal{D} with the difference that the suprema are now taken over C instead of over R^n . It is easy to see that $\mathcal{D}(C)$ is a complete space and that the restriction operator \mathcal{P} is continuous. Let \mathcal{P}^* be its adjoint so that

$$\langle \mathcal{P}^* f, g \rangle = \langle f, \mathcal{P} f \rangle,$$

where $f \in \mathcal{D}(C)^*$ and $\mathcal{P}^* f \in \mathcal{D}^*$. If $g = 0$ on C , then $\langle \mathcal{P}^* f, g \rangle = 0$, so that $\mathcal{P}^* \mathcal{D}(C)^*$ is contained in the set \mathcal{D}_C^* of all $F \in \mathcal{D}^*$ whose supports are contained in C . Under a certain regularity condition on C , which holds whenever C is a finite union of convex sets, it can be shown that \mathcal{P}^* is a linear homomorphism between $\mathcal{D}(C)^*$ and \mathcal{D}_C^* . This makes it possible to identify these two spaces. We shall use this result later when C is the region $x_1^2 - x_2^2 \dots x_n^2 \geq 0$, i.e. a union of two circular cones.

6.10. Products of δ -functions. — Let g be a real function. What should we mean by $\delta(g(x))$?

It is of course natural to put

$$\langle \delta(g), h \rangle = \lim_{\varepsilon \rightarrow 0} \int \varphi(g(x)) h(x) dx,$$

where $\varphi = \varphi_\varepsilon \in \mathcal{D}(R)$ approximates δ as $\varepsilon \rightarrow 0$. If $g \in \mathcal{E}(\mathcal{O})$ and its gradient does not vanish in a neighborhood of $g = 0$, we can introduce new co-ordinates

y such that $y_1 = g(x)$ and then

$$\lim_{\varepsilon \rightarrow 0} \int \varphi(y_1) h(x) p(y) dy = \int [h(x) p(y)]_{y_1=0} dy_2 \dots dy_n,$$

where $dx = p(y) dy$, so that the limit exists and defines a distribution.

Exercise. - It is of course possible that the definition works also when $\text{grad } g = 0$ in some points. Verify this when $g = x_1^2 - x_2^2 - \dots - x_n^2$.

Exercise. - Show that we can define $\delta(g_1) \dots \delta(g_m)$ when g_1, \dots, g_m can be chosen as m of the co-ordinates in a co-ordinate system covering a neighborhood of $g_1 = \dots = g_m = 0$.

Exercise. - Let p be a real polynomial with only simple real zeros x_k . Show that

$$\delta(p(x)) = \sum p'(x_k)^{-1} \delta(x - x_k).$$

Exercise. - Show that $\delta(|x|) = \delta$ and $\delta(|x|^{\frac{1}{2}}) = 0$ and that $\delta(x^2)$ does not define a distribution.

6.11. *General principal values.* - Let $g \in \mathcal{E}'(\mathcal{O})$ be zero on a hypersurface of dimension $n - 1$. We define Pg^{-1} as

$$\langle Pg^{-1}, h \rangle = \lim_{\varepsilon \rightarrow 0} \int_{|g(x)| \geq \varepsilon} \frac{h(x)}{g(x)},$$

when the limit exists and defines a distribution.

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7.1. - Fourier transform (J. L. LIONS).

1) We recall the definition of \mathcal{S} :

$$f \in \mathcal{S} \Leftrightarrow \left\{ \begin{array}{l} f \text{ infinitely differentiable in } R^n, \\ |x_\alpha D^\beta f(x)| \leq M_{\alpha\beta} \text{ for every } \alpha \text{ and } \beta \Leftrightarrow \\ \Leftrightarrow \left\{ \begin{array}{l} f \text{ infinitely differentiable,} \\ |x_\alpha D^\beta f(x)| \rightarrow 0 \text{ when } |x| \rightarrow \infty, \text{ for every } \alpha \text{ and } \beta. \end{array} \right. \end{array} \right.$$

Examples: $\exp[-\pi x^2]$, $\exp[-x^{2n}]$ (n integer > 0) $\in \mathcal{S}$.

Definition of Fourier transform onto \mathcal{S} :

$$\mathcal{F}f(x) = \hat{f}(x) = \int \exp[-2\pi ixy] f(y) dy ;$$

if $f \in \mathcal{S}$, this is obviously meaningful.

Proposition 1. $\mathcal{F}(D^p f) = (2\pi i x)_p (\mathcal{F}f)$, $f \in \mathcal{S}$.

Proposition 2. $\mathcal{F}((-2\pi i x)_p f) = D^p \hat{f}$, $f \in \mathcal{S}$.

By using these two results, one easily proves:

Proposition 3. If $f \in \mathcal{S}$ then $\hat{f} \in \mathcal{S}$.

Moreover, the mapping $f \rightarrow \hat{f}$ is continuous of \mathcal{S} into \mathcal{S} .

2) We now introduce \mathcal{S}^* = dual space of \mathcal{S} = space of tempered distributions.

Exercise. - \mathcal{S}^* is invariant by differentiation, by translation.

We know (Theorem 5'9) the general structure of elements of \mathcal{S}^* . We recall also that \mathcal{S} is dense into \mathcal{S}^* .

LEMMA 1. - The mapping $f \rightarrow \mathcal{F}f$ is continuous from \mathcal{S} to \mathcal{S} , when both spaces are provided with the topology induced by \mathcal{S}^* .

Proof. - As is easily checked, if f and $\varphi \in \mathcal{S}$, we have

$$(7.1.1) \quad \langle \mathcal{F}f, \varphi \rangle = \langle f, \mathcal{F}\varphi \rangle .$$

Now, if $f_n \rightarrow 0$ in \mathcal{S}^* , I claim that $\mathcal{F}f_n \rightarrow 0$ into \mathcal{S}^* . Indeed, assume that $\varphi \in B =$ bounded set into \mathcal{S} ; then $\mathcal{F}\varphi$ belongs to a bounded set \hat{B} into \mathcal{S} (Proposition 3), and then

$\langle f_n, \mathcal{F}\varphi \rangle \rightarrow 0$ uniformly for $\varphi \in B$; Lemma 1 follows by using (7.1.1).

From Lemma 1 it follows that the mapping $f \rightarrow \mathcal{F}f$ can be extended by continuity to a continuous linear mapping $T \rightarrow \mathcal{F}T$ from \mathcal{S}^* into itself. Equation (7.1.1) leads to

$$(7.1.2) \quad \langle \mathcal{F}T, \varphi \rangle = \langle T, \mathcal{F}\varphi \rangle , \quad T \in \mathcal{S}^*, \varphi \in \mathcal{S} .$$

Summing up:

THEOREM 7.1.1. - The transformation \mathcal{F} can be extended by continuity to a continuous linear mapping from \mathcal{S}^* into itself; if $T \in \mathcal{S}^*$, $\mathcal{F}T$ can be calculated with (7.1.2).

Exercise. - Prove that $\mathcal{F}(\delta) = 1$.

By using Propositions 1 and 2 and Lemma 1, one gets:

Proposition 4. For every $T \in \mathcal{S}^*$:

$$\left. \begin{aligned} \mathcal{F}(D^p T) &= (2\pi i x)_p \mathcal{F}T; \quad \mathcal{F}((-2\pi i x)_p T) = D^p(\mathcal{F}T); \\ \mathcal{F}(T(x-a)) &= \exp[-2\pi i a x] \mathcal{F}T; \quad \mathcal{F}(\exp[2\pi i a x] T) = (\mathcal{F}T)(x-a). \end{aligned} \right\} \text{Diagonalization}$$

Exercise. — If $T(x) = T_1(x_1) T_2(x_2) \dots T_n(x_n)$, $T_i \in \mathcal{S}^*(R)$, then $\mathcal{F}T(x) = (\mathcal{F}T_1(x_1)) \dots (\mathcal{F}T_n(x_n))$.

\mathcal{F} is n dimensional in the left side, and is one dimensional in the right side.

Transformation $\overline{\mathcal{F}}$:

$$\overline{\mathcal{F}}f(x) = \int \exp[2\pi i xy] f(y) dy, \quad f \in \mathcal{S}.$$

By the same reasoning as above, one proves that $\overline{\mathcal{F}}$ can be extended by continuity to a continuous linear mapping from \mathcal{S}^* into itself.

Exercise. — $\overline{\mathcal{F}}(\exp[-\pi x^2]) = \exp[-\pi x^2]$; $\overline{\mathcal{F}}(\delta) = 1$.

7.1.3. — *The fundamental Lemma:* $\mathcal{F}(\exp[2\pi i ax]) = \delta(x-a)$ (Dirac mass in the point $a \in R^n$).

By Proposition 4, it is enough to prove

$$(7.1.3) \quad \mathcal{F}(1) = \delta.$$

Using the exercise on $T(x) = T_1(x_1) \dots T_n(x_n)$, applied to $\delta(x) = \delta(x_1) \dots \delta(x_n)$ it is enough to prove (7.1.3) in dimension 1. We set: $\mathcal{F}(1) = T$. By using Proposition 4, we get:

$$\mathcal{F}\left(\frac{d}{dx} 1\right) = 0 = 2\pi i x T,$$

i.e. $xT = 0$, hence (Exercise, Sect. 6) $T = c\delta$. It remains to calculate c ; we apply (7.1.2):

$$\langle c\delta, \varphi \rangle = \langle 1, \mathcal{F}\varphi \rangle,$$

and choose $\varphi = \exp[-\pi x^2]$, which equals its Fourier Transform. Then $c = 1$ and the fundamental Lemma is proved.

Exercise. — Prove the same lemma using $\mathcal{F}(\exp[-\pi x^2/n^2]) = n \exp[-\pi x^2 n^2]$ and assuming that $n \rightarrow \infty$.

7.1.4. *The fundamental Theorems.* — Take $f \in \mathcal{S}$; of course $f(a) = \langle \delta_{(a)}, f \rangle = \langle \mathcal{F}(\exp[2\pi i ax]), f \rangle$ by using the Fundamental Lemma.

Using now (7a.2), $f(a) = \langle \exp[2\pi i ax], \mathcal{F}f \rangle$, i.e. since a is arbitrary:

$$f(y) = \int \exp[2\pi i xy] \hat{f}(x) dx,$$

i.e.

$$f = \overline{\mathcal{F}}(\mathcal{F}f), \quad f \in \mathcal{S}.$$

We have proved:

THEOREM 7'1.2. - If $f \in \mathcal{S}$, $\hat{f} = \mathcal{F}f$, then

$$f = \overline{\mathcal{F}}\hat{f} \quad (\text{Fourier inversion formula}).$$

One can also write:

$$\mathcal{F}\overline{\mathcal{F}} = 1_{\mathcal{S}}, \quad \overline{\mathcal{F}}\mathcal{F} = 1_{\mathcal{S}}, \quad 1_{\mathcal{S}} = \text{identity mapping into } \mathcal{S}.$$

Also: \mathcal{F} is an isomorphism from \mathcal{S} onto \mathcal{S} , with inverse $\overline{\mathcal{F}}$.

All these results extend by continuity to \mathcal{S}^* , hence the main result:

THEOREM 7'1.3. - \mathcal{F} is an isomorphism of \mathcal{S}^* onto itself, with inverse $\overline{\mathcal{F}}$.

7'1.5. *Fourier transform on L^2* . - Take once more $f \in \mathcal{S}$; we want to calculate $\int |f(x)|^2 dx = \|f\|^2$; $\|f\|^2 = \langle f, \bar{f} \rangle = \langle \mathcal{F}\hat{f}, \bar{f} \rangle = \langle \bar{f}, \mathcal{F}\bar{f} \rangle$ hence easily

LEMMA 2. - If $f \in \mathcal{S}$, $\int |f(x)|^2 dx = \int |\hat{f}(x)|^2 dx$. In other words:

$$\|f\| = \|\hat{f}\|.$$

This means that the mapping $f \rightarrow \hat{f}$ is continuous from \mathcal{S} provided with the topology induced by L^2 into L^2 . We admit here that \mathcal{S} is dense in L^2 . Then

THEOREM 7'1.4. - The mapping $f \rightarrow \hat{f}$ from $\mathcal{S} \rightarrow \mathcal{S}$ can be extended by continuity into a continuous linear mapping \mathcal{F} of L^2 into L^2 ; actually, \mathcal{F} is an isomorphism of L^2 onto itself, with inverse $\overline{\mathcal{F}}$. Moreover:

$$\|f\| = \|\mathcal{F}f\|.$$

Remark 1. - If $f \in L^2$, then in particular $f \in \mathcal{S}^*$; we define the same $\mathcal{F}f$ by considering f as an element of L^2 or \mathcal{S}^* .

Remark 2. - If $f \in L^2$,

$$\hat{f}(x) = \text{l.i.m.} \int_{R \rightarrow \infty} \exp[-2\pi ixy] f(y) dy,$$

l.i.m. = limit in the mean.

Exercise. - If $\int |f(x)| dx < \infty$, \hat{f} is bounded.

Exercise. - $f \in \mathcal{S} \Leftrightarrow (1+x^2)^p D^q f \in L^2$ for every p and q .

7.1.6. *Paley Wiener theorem.* — We admit here the following theorem:

THEOREM 7.1.5. — Let T be in \mathcal{S}^* ; the two following statements are equivalent:

1) T has a compact support, $\subset [-A, A]^n$;

2) \hat{T} is a function $\hat{T}(x)$, with $|\hat{T}(x)|$ bounded by a polynomial, and $T(x)$

is the restriction to R^n of a function $F(z)$, $z = (z_1, \dots, z_n)$, $z_j = x_j + iy_j$, $F(z)$ being entire, with

$$|F(z)| \leq M \exp [2\pi A(|z_1| + |z_2| + \dots + |z_n|)] .$$

That 1) implies 2) is quite simple. That 2) \Rightarrow 1) goes deeper; it rests essentially on Phragmen Lindelof theorem.

7.1.7. *Examples.* — Let $\mu_{(a)}$ be the measure defined by:

$$\langle \mu_{(a)}, f \rangle = \text{mean value of } f \text{ onto the sphere } |x| = a .$$

One has:

$$\mathcal{F}\mu_{(a)} = \Gamma(n/2)(\pi a |x|)^{(n-2)/2} J_{(n-2)/2}(2\pi a |x|) .$$

Application: let $T(x)$ be a function or a distribution $\in \mathcal{S}^*$, which depends only on $|x| = r$; $T(x) = U(r)$. Then

$$\int \exp[-2\pi ixy] T(y) dy = \int_0^\infty U(\varrho) d\varrho \int_{|y|=P} \exp[-2\pi ixy] d\sigma_y ,$$

where $d\sigma_y$ = surface area onto $|y| = \varrho$. An easy calculation leads to

$$\hat{T}(x) = V(r) = 2\pi r^{(n-2)/2} \int_0^\infty \varrho^{n/2} J_{(n-2)/2}(2\pi \varrho r) U(\varrho) d\varrho ,$$

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SCHWARTZ: *Théorie des distributions* - t. II.

7.2. — Convolution and multiplication with Fourier transform (J. L. LIONS).

7.2.1. *Some more properties of \mathcal{S} and \mathcal{S}^* . The problems.*

a) if $f, g \in \mathcal{S}$, then $fg \in \mathcal{S}$ and $\mathcal{F}(fg) = \hat{f} * \hat{g}$. It follows: if $f, g \in \mathcal{S}$, $f * g \in \mathcal{S}$.

Exercise. — Prove directly this last statement.

We have

$$(7.2.1) \quad \mathcal{F}(f * g) = \hat{f} \hat{g} .$$

b) If $f \in \mathcal{S}$, $T \in \mathcal{S}^*$, then $f * T \in \mathcal{S}^*$.

c) If $f \in \mathcal{S}$, $T \in \mathcal{S}^*$, $fT \in \mathcal{S}^*$ and

$$(7'2.2) \quad \mathcal{F}(fT) = \hat{f} * \hat{T}.$$

d) The properties a), b), c) lead in natural way to the following problems:

I) What is the space, say \mathcal{O}_M , of the functions $f \in \mathcal{E}$ such that $fT \in \mathcal{S}^*$ for every $T \in \mathcal{S}^*$?

II) What is the space, say \mathcal{O}'_c , of the distributions S such that $S * T$ is defined and $S * T \in \mathcal{S}^*$?

III) Is it true that \mathcal{F} is an isomorphism from \mathcal{O}_M onto \mathcal{O}'_c , and is it true that

$$\mathcal{F}(S * T) = \hat{S} \hat{T}, \quad S \in \mathcal{O}'_c, \quad T \in \mathcal{S}^*?$$

7'2.2. — *Space \mathcal{O}_M .* — Let a be in \mathcal{O}_M . If $f \in \mathcal{S}$, then af is necessarily into \mathcal{S} (since $\langle af, T \rangle = \langle f, aT \rangle$, $T \in \mathcal{S}^*$, and $aT \in \mathcal{S}^*$ by hypothesis). By using the closed graph theorem, we see that

$$(7'2.3) \quad f \rightarrow af$$

is a *continuous* linear mapping from \mathcal{S} into itself. (*Remark:* we can escape the use of the closed graph theorem by imposing the condition that $T \rightarrow aT$ is continuous of \mathcal{S}^* into itself for every $a \in \mathcal{O}_M$).

We write now that (7'2.3) is continuous, with explicit neighborhoods of 0 in \mathcal{S} . We denote by $V_{k,m}^{\mathcal{E}}$ the set of $f \in \mathcal{S}$, such that

$$(1 + x^2)^k |D^p f(x)| \leq \mathcal{E}, \quad |p| \leq m;$$

this gives a fundamental set of neighborhoods of 0 in \mathcal{S} . Since (7'2.3) is continuous, given $V_{0,0}^1$ there exist k, m, \mathcal{E} such that

$$(7'2.4) \quad af \in V_{k,m}^{\mathcal{E}} \quad \text{for every } f \in V_{0,0}^1.$$

We set: $\mathcal{S}_{k,m}$ = functions f such that $(1 + x^2)^k |D^p f(x)|$, $|p| \leq m$, is zero at infinity; it is a Banach space for the norm

$$\sup_{|p| \leq m, x} (1 + x^2)^k |D^p f(x)|.$$

But (7'2.4) means: $f \rightarrow af$ is continuous of \mathcal{S} provided with the topology induced by $\mathcal{S}_{k,m}$ into $\mathcal{S}_{0,0}$. As it is easily verified \mathcal{S} is dense in $\mathcal{S}_{k,m}$; conclu-

sions: (7.2.3) can be extended as a continuous linear mapping of $\mathcal{S}_{k,m}$ into $\mathcal{S}_{0,0}$. But $f(x) = (1+x^2)^{-k-1} \in \mathcal{S}_{k,m}$ so that $a(x)(1+x^2)^{-k-1}$ is bounded. We have proved:

LEMMA 1. — If $a \in \mathcal{O}_M$, then there exist k' and $M_{k'}$ such that

$$(7.2.5) \quad |a(x)| \leq M_{k'}(1+x^2)^{k'}.$$

Now, if $aT \in \mathcal{S}^*$, $(\partial/\partial x_i)(aT)$ belongs to \mathcal{S}^* too, hence: $((\partial/\partial x_i)a)T \in \mathcal{S}^*$, i.e. $(\partial/\partial x_i)a \in \mathcal{O}_M$, and finally: $D^p a \in \mathcal{O}_M$ for every p . Then Lemma 1 implies:

$$a \in \mathcal{O}_M \Rightarrow |D^p a(x)| \leq \text{polynomial (depending on } p), \text{ for every } p.$$

It is very simple to prove that these conditions are sufficient to insure $a \in \mathcal{O}_M$; we can state:

THEOREM 7.2.1. — The following two statements are equivalent.:

- 1) $a \in \mathcal{O}_M$;
- 2) $|D^p a(x)| \leq \text{polynomial } P_p(x) \text{ for every } p$.

Remark 1. — $\mathcal{L}(\mathcal{S}; \mathcal{S})$ = space of continuous linear mappings from $\mathcal{S} \rightarrow \mathcal{S}$.

For every $a \in \mathcal{O}_M$ we set: $\tilde{a} \in \mathcal{L}(\mathcal{S}; \mathcal{S})$, continuous linear mapping $f \rightarrow af$. We define in this way a mapping $a \rightarrow \tilde{a}$ from \mathcal{O}_M into $\mathcal{L}(\mathcal{S}; \mathcal{S})$. This mapping is one to one. Then we can identify \mathcal{O}_M with a subspace of $\mathcal{L}(\mathcal{S}; \mathcal{S})$, and in this way we can put on \mathcal{O}_M a natural topology: the topology induced by the topology of $\mathcal{L}(\mathcal{S}; \mathcal{S})$.

Exercise. — Let $P(x)$ be a polynomial with real coefficients. Then $\exp[iP(x)] \in \mathcal{O}_M$.

Exercise. — If $S \in \mathcal{S}^*$, $f \in \mathcal{S}$, then $S*f \in \mathcal{O}_M$.

7.2.3. Space \mathcal{O}'_c . — Let S be in \mathcal{O}'_c : take $T \in \mathcal{E}^*$, $f \in \mathcal{D}$; by definition

$$\langle S*T, f \rangle = \langle T(x), \langle S(y), f(x+y) \rangle \rangle.$$

We want that this equation has a meaning when $T \in \mathcal{S}^*$, $f \in \mathcal{S}$, i.e.

$$\langle S(y), f(x+y) \rangle \in \mathcal{S}.$$

Obviously, this is equivalent to: $S*f \in \mathcal{S}$ for every $f \in \mathcal{S}$, hence:

THEOREM 7.2.2. — The two following statements are equivalent:

- 1) $S \in \mathcal{O}'_c$;
- 2) $S*f \in \mathcal{S}$ for every $f \in \mathcal{S}$ (this implies $S \in \mathcal{S}^*$).

Exercise. — 1 (which belongs to \mathcal{S}^*) is not in \mathcal{O}'_o .

Example: $\exp[i\pi x^2] \in \mathcal{O}'_o$: take $\exp[i\pi x^2] * f$, and use 2).

7'2.4. Relations between \mathcal{O}_M and \mathcal{O}'_o . — We can write:

$S \in \mathcal{O}'_o \Leftrightarrow S * f \in \mathcal{S}$ for every $f \in \mathcal{S} \Leftrightarrow \hat{f} \hat{S}$ for every $\hat{f} \in \mathcal{S} \Leftrightarrow \hat{S} \in \mathcal{O}_M$. Then:

THEOREM 7'2.3. — \mathcal{F} is an isomorphism of \mathcal{O}'_o onto \mathcal{O}_M .

Exercise. — $\mathcal{F}(\exp[i\pi x^2]) = \left(\frac{1+i}{\sqrt{2}}\right) \exp[-i\pi y^2]$.

Exercise. — If $f \in \mathcal{S}$, $T \in \mathcal{S}^*$, then $fT \in \mathcal{O}'_o$ (use Fourier transform, Theorem 7'2.3 and an exercise in 7'2.2).

7'2.5. Exchange of convolution and multiplication.

THEOREM 7'2.4. — If $S \in \mathcal{O}'_o$, $T \in \mathcal{S}^*$, we have

$$(7'2.6) \quad \mathcal{F}(S * T) = (\mathcal{F}S)(\mathcal{F}T).$$

Proof. — We approach T by a sequence T_n of elements of \mathcal{S} ; for T_n , the equation corresponding to (7'2.6) is true, hence the general case by passing to the limit.

All of the problems of 1) are solved.

Exercise. — If $S, T \in \mathcal{O}'_o$, then $S * T \in \mathcal{O}'_o$.

Exercise. — If $S, T \in \mathcal{O}'_o$, and $S * T = 0$, is it true that either S or T is 0? (the answer is no).

7'2.6. Applications (I). — A convolution equation is an equation of the form:

$$(7'2.7) \quad A * T = S, \quad A \text{ given in } \mathcal{O}'_o, \quad S \text{ given in } \mathcal{S}^*; \text{ we look for } T \text{ in } \mathcal{S}^*.$$

Example:

(7'2.8) $A = P\delta$, P = partial differential operator with constant coefficients. Then $A \in \mathcal{E}^* \subset \mathcal{O}'_o$. In that case, (7'2.7) is written simply $PT = S$.

By Fourier transform and the above theorems, (7'2.7) is equivalent with

$$(7'2.9) \quad A(x)T(x) = S(x), \quad A(x) \in \mathcal{O}_M.$$

It is a *division problem*.

By a Theorem of Hörmander and Lojasiewicz already stated, we know that equation (7'2.9) has always at least one solution.

Exercise. — $A = \partial^2/\partial x_1^2 + \dots + \partial^2/\partial x_M^2$; prove that $(-A+1)T = S$ admits one and only one solution in \mathcal{S}^* . If $S \in \mathcal{S}^*$, and if we look for T in \mathcal{D}^* , then of

course we have existence of a solution, but the solution is not unique (take exponential solutions of $(-\Delta + 1)u = 0$). In general, if $S \in (\mathcal{D}^*)$, then it can be proved that there exists $T \in \mathcal{D}^*$ with $PT = S$. (MALGRANGE; EHRENFREIS).

7.2.7. *Applications* (II). — Let t be the time, $t \geq 0$. We give a family of distributions $A(x; t)$, on R^n , and we assume that for every $t \geq 0$, $A(x; t)$ is a distribution with compact support. (Actually, in the main applications, $A(x; t)$ has the origin for support; $A(x; t)^*$ is a differential operator).

We look for $u(x, t)$, $u(x, t) \in \mathcal{S}^*$ for every $t \geq 0$, solution of

$$\frac{\partial}{\partial t} u + A(x; t)^* u(x, t) = 0, \quad u(x, 0) = f(x), \quad f \in \mathcal{S}^* \text{ given.}$$

(To be more precise: we look for a function $t \rightarrow u(\cdot, t)$, once continuously differentiable for $t \geq 0$, with values into \mathcal{S}^* , solution of the above equations.)

By Fourier transform, this problem is equivalent to

$$\frac{\partial}{\partial t} v + A(y; t)v(y, t) = 0, \quad v(y, t) = f(y).$$

If we consider y as a parameter, we obtain an ordinary differential equation and then $v(y, t)$ is known (of course, one has to be slightly careful: $f(y)$ is not a function in general). Now, if $v(y; t) \in \mathcal{S}^*$ for every t , then

$$u(x, t) = \int \exp[2\pi ixy] v(y, t) dy$$

is the solution of the problem.

Remark. — Replacing $A(x, t)$ by $\|A_{ij}(x, t)\|$ the method applies to differential equations of any order in t .

Exercise. — To solve by the above method the following two classical problems:

$$(1) \quad \frac{\partial}{\partial t} u - \frac{\partial^2}{\partial x^2} u = 0, \quad u(x, 0) = f(x);$$

$$(2) \quad \frac{\partial^2}{\partial t^2} u - \frac{\partial^2}{\partial x^2} u = 0, \quad u(x, 0) = f(x),$$

$$\frac{\partial}{\partial t} u(x, 0) = g(x).$$

LITERATURE

SCHWARTZ: *Théorie des distributions*, t. II.

8. - Lorentz invariant distributions (L. GÄRDING).

Let $h \in \mathcal{D}(R^4)$ and let A be a Lorentz transformation. We define the action of A on h by putting

$$(Ah)(x) = h(A^{-1}x)$$

and the action of A on a distribution f by putting

$$\langle Af, h \rangle = \langle f, A^{-1}h \rangle.$$

Since $\int f(A^{-1}x)h(x)dx = \int f(x)h(Ax)dx$, the two actions are the same when f is a function. We say that f is invariant if $Af = f$ for every A that does not reverse time and we are going to determine all such f and study their Fourier transforms. In the first place, let A_0 be a Lorentz transformation which reverses time and put

$$f_1 = \frac{1}{2} (f + A_0 f),$$

$$f_2 = \frac{1}{2} (f - A_0 f).$$

Then, for an arbitrary A we have

$$Af_1 = f_1, \quad Af_2 = \varepsilon(A)f_2,$$

where $\varepsilon(A) = \pm 1$ according as A reverses time or not. Let us refer to the corresponding distributions as even and odd respectively. The corresponding sets of distributions will be called \mathcal{L}^+ and \mathcal{L}^- respectively. We are going to treat them separately.

Let us start with a rough heuristic argument. Let $f \in \mathcal{L}^+$. Treating it as a function, we see that it ought to be constant on the hypersurfaces $xx = x^0x^0 - x^1x^1 - x^2x^2 - x^3x^3 = c$, also when $c \geq 0$, and the surface consists of a cone or a hyperboloid with two sheets. Hence

$$(1) \quad f(x) = \varphi(xx),$$

where φ is a function of one variable, defined on the whole real axis. Conversely, any such distribution seems to be even and invariant. The argument has the disadvantage that it does not seem to give the Lorentz invariant distributions with supports at the origin, namely

$$(2) \quad f = P(\square)\delta,$$

where δ is the four-dimensional δ -function, $\square = D_0 D_0 - D_1 D_1 - D_2 D_2 - D_3 D_3$, ($D_k = \partial/\partial x^k$) is the wave operator and P is any polynomial. A similar heuristic argument shows that every f in \mathcal{E}' ought to have the form

$$(3) \quad f(x) = \varphi(xx) \operatorname{sgn} x_0,$$

where φ is a distribution of one variable which vanishes when $xx < 0$. This last property can be justified as follows. The distribution f changes its sign under Lorentz reflections in any space-like plane. Hence it ought to vanish on such planes, *i.e.* outside the cone $xx \geq 0$.

Let us now try to justify (1). Define

$$(Mg)(\tau) = \int g(x) \delta(\tau - xx) dx.$$

It is an even invariant distribution whose support is the hypersurface $xx = \tau$. When $\tau < 0$, this is a hyperboloid with one sheet, when $\tau = 0$, it is the light-cone and when $\tau > 0$, it is a hyperboloid with two sheets.

Let Q be R^4 with the origin excluded.

LEMMA 8'1. — To every invariant even distribution f , there is a unique $\varphi \in \mathcal{D}^*(R)$ such that

$$(4) \quad \langle f, g \rangle = \langle \varphi, Mg \rangle, \quad g \in \mathcal{D}(Q)$$

and, for any φ , the right side is an even invariant distribution in $\mathcal{D}(Q)^*$.

Proof. — Any Lorentz invariant distribution f has the property that

$$\langle f, A_{jk}g \rangle = 0 \quad \text{where} \quad A_{jk} = x_j D_k - x_k D_j, \quad (x = x^0, x_k = -x^k, k > 0),$$

are the infinitesimal Lorentz transformations. Now every point $y \neq 0$ has a convex open neighborhood \mathcal{O} not containing 0, where one of the co-ordinates, say x_j , does not vanish. In \mathcal{O} we take τ as a new co-ordinate instead of x_j . Then

$$\delta_k = x_j^{-1} A_{jk}, \quad (j \neq k),$$

where the right side denotes differentiation with respect to x^k in the new system. Consequently, if $f_1 = 2f|x_j|$, we get

$$\langle f_1, \delta_k g \rangle = \pm 2 \langle f, x_j \delta_k g \rangle = \pm 2 \langle f, A_{jk} g \rangle = 0,$$

when $k \neq j$ and $g \in \mathcal{D}(\mathcal{O})$. Hence, if $I = I(\mathcal{O})$ is the range of xx when $x \in \mathcal{O}$,

there is a unique φ in $\mathcal{D}^*(I)$ such that

$$\langle f_1, g \rangle = \langle \varphi, Ng \rangle,$$

where

$$(Ng)(\tau) = \int g(x) d\omega, \quad d\omega = dx^0 \dots \widehat{dx^j} \dots$$

But, since

$$(Mg)(\tau) = \int g(x) \frac{d\omega}{2|x^j|}, \quad xx = \tau,$$

this means that

$$\langle f, g \rangle = \langle \varphi, Mg \rangle, \quad g \in \mathcal{D}(\mathcal{O}).$$

Now, let \mathcal{O} and \mathcal{O}' be two neighborhoods of the kind we have described. Performing a Lorentz transformation A we see that φ is unique in $I(A\mathcal{O} \cap \mathcal{O}')$. Since the whole Lorentz group acts transitively on Q , it follows from this that φ is independent of \mathcal{O} and hence the lemma follows.

LEMMA 8'2. — Any invariant distribution f whose support is the origin has the form (2) and conversely.

Proof. — We know that f must be a linear combination of derivatives of δ ,

$$f = A(D)\delta,$$

where $A = A(D_0, \dots, D_3)$ is a polynomial. That f is invariant means that A is invariant. An easy argument shows that A must have the form $P(\square)$.

Our Lemma does not quite give us all invariant distributions yet since Lemma 1 applies only to $\mathcal{D}(Q)$. The complete theory has to involve a description of the space $M\mathcal{D}$, i.e. the image of $\mathcal{D} = \mathcal{D}(R^4)$ under the mean value M .

Let R be the real line and consider the set $H = H(R)$ of all complex functions on R which are infinitely differentiable outside the origin, vanish for large arguments and have the following property. To every $h \in H$ there is a formal power series Lh with partial sums

$$(L_n h)(\tau) = \sum_{k=0}^n \langle l_k, h \rangle \tau^k, \quad \langle l_0, h \rangle = 0, \text{ all } h,$$

such that, for all $n \geq 0$,

$$h(\tau) - L_n h(\tau) \log |\tau|^{-1}$$

is n times continuously differentiable at the origin.

Exercise. — Show that Lh is uniquely determined by h and that L and the l_k are linear functions of h . We shall call Lh the singular part of h .

We shall see later that $M\mathcal{D} = H$, but first we shall have a closer look at H . It is clear that if $h_1, h_2 \in \mathcal{D}(R)$ and $h_2(0) = 0$, then

$$(5) \quad h(\tau) = h_1(\tau) + h_2(\tau) \log |\tau|^{-1}$$

belongs to H . The power series Lh is in this case the Taylor series of h_2 at the origin. Conversely, let $h \in H$. Then, by a classical theorem due to Borel there is an $h_2 \in \mathcal{D}(R)$ with the Taylor series Lh . But then, if (5) is taken as a definition of h_1 , it follows that $h_1 \in \mathcal{D}(R)$ and consequently (5) is the general form of a function in H .

Let us now topologize H . Let $\varphi \in \mathcal{D}(R)$ be a fixed function which is 1 in a neighborhood of the origin and vanishes when $|\tau| \geq \frac{1}{2}$. Let p be any seminorm on $\mathcal{D}(R)$ and choose a number n such that the order of p is $\leq n$ in the interval $|\tau| \leq 1$. Put

$$g(h) = p(h - \log |\tau|^{-1} \varphi L_n h) + \sum_0^n |\langle l_k, h \rangle|.$$

These seminorms define a topology on H which is independent of the choice of φ and reduces to the usual topology on $\mathcal{D}(R)$. It is easy to see that H is complete.

LEMMA 8'3. — Every continuous linear functional F on H has the form

$$\langle F, h \rangle = \langle f, h - \log |\tau|^{-1} \varphi L_n h \rangle + \sum c_k \langle l_k, h \rangle,$$

where $f \in \mathcal{D}(R)^*$ has order $\leq n$ in the interval $|\tau| \leq 1$ and the numbers c_k are arbitrary.

Proof. — On $\mathcal{D}(R)$, F reduces to an element $f \in \mathcal{D}(R)^*$. Let n exceed the order of f in $|\tau| \leq 1$ and put

$$\langle F_1, h \rangle = \langle f, h - \log |\tau|^{-1} \varphi L_n h \rangle.$$

Then $F_1 \in H^*$ and $\langle F - F_1, h \rangle = 0$ when $L_n h = 0$. Hence $F - F_1 = \sum c_k l_k$ and the Lemma follows.

Exercise. — Every $h \in H$ has an asymptotic development

$$h \sim \sum (\langle l_k, h \rangle \log |\tau|^{-1} + j_k k) \tau^k, \quad l_0 = 0,$$

for small τ . Show that the l_k and j_k form a basis for those $F \in H^*$ which vanish outside the origin.

Exercise. — Let $p \neq 0$ be a polynomial. Show that the mapping $H \ni h \rightarrow ph \in H$ has a continuous inverse. (Hint. Suffices to consider $p = \tau - \lambda$. When λ is not real, the statement is trivial. The case λ real $\neq 0$, is essentially the statement that $\mathcal{D}(R) \ni h \rightarrow ph \in \mathcal{D}(R)$ has a continuous inverse, which has been proved earlier. The case $\lambda = 0$ can be treated in a similar way.) Use this to show that $pH^* = H^*$.

Exercise. — Show that $M \square g(\tau) = 4\Gamma M g(\tau)$, where $\Gamma = \tau D^2$, ($D = d/d\tau$), and show that $H \ni h \rightarrow \Gamma h \in H$ is continuous. Let $\Gamma^* = D^2 \tau$ be the adjoint of Γ . Show that

$$\Gamma^* l_k = k(k+1)l_{k+1}, \quad \Gamma^* j_k = k(k+1)j_{k+1} - (2k+1)l_{k+1}.$$

In particular, $\Gamma^* j_0 = -l_1$.

Exercise. — Show that the mapping $H \ni h \rightarrow \Gamma h \in H$ has a continuous inverse. (If $g = \Gamma h$, then $h(\tau) = \int_0^\tau (\tau \sigma^{-1} - 1)g(\sigma) d\sigma$. Deduce the expansion of h at the origin from that of g .)

Exercise. — Show that $P\tau^{-1}$ and 1 are a basis for all solutions $F \in H^*$ of $\Gamma^* F = 0$. (Show that $F = c_0 + c_1 \tau^{-1}$ outside the origin and that the constants c_0 and c_1 are the same for $\tau > 0$ and $\tau < 0$. This means that $F = c_0 P\tau^{-1} + c_1 F_0$ where the support of F_0 is the origin. Show that $F = 0$. By the same method one gets a basis for the solutions F of $\Gamma F = \lambda F$. Depending on λ , they may or may not be tempered, *i.e.* belong to $(MS)^*$ (METHÉE).)

Let us now prove

LEMMA 8'4. — The mean value M induces continuous linear mapping of $\mathcal{D}(Q)$ and $\mathcal{D}(R^*)$ onto $\mathcal{D}(R)$ and H respectively.

Proof. — It is clear that if \mathcal{O} is a small convex neighborhood of any point y in Q , then $M\mathcal{D}(\mathcal{O}) \subset \mathcal{D}(R)$ and we leave it to the reader to prove that M is continuous and that $M\mathcal{D}(\mathcal{O})$ contains $\mathcal{D}(I(\mathcal{O}))$. Since the $I(\mathcal{O})$ cover R , the first part of the Lemma follows.

The second part of the Lemma requires a little calculation. Consider an auxiliary transformation. A defined by

$$Ag(\varrho, \sigma) = \frac{1}{4} \int (g(\sqrt{\varrho}, \sqrt{\sigma} \omega) + g(-\sqrt{\varrho}, \sqrt{\sigma} \omega) d\omega,$$

where $\varrho \geq 0$, $\sigma \geq 0$, ω ranges over the unit sphere in three dimensions with the element of area $d\omega$. Clearly, A maps $\mathcal{D}(R^+)$ continuously into $\mathcal{D}(R^+ \times R^+)$, where R^+ is the closed positive half-axis. That the mapping is onto $\mathcal{D}(R^+ \times R^+)$ is easy to prove. In fact, if $h \in \mathcal{D}(R^+ \times R^+)$ then $2\pi g(x) = h(x_0^2, x_1^2 + x_2^2 + x_3^2)$ is in

$\mathcal{D}(R^4)$ and $Ag = h$. Taking $\bar{x} = (x^1, x^2, x^3)$ as variables on $xx = \tau$ we get

$$Mg(\tau) = \int (2x^0)^{-1} (g(x^0, \bar{x}) + g(-x^0, \bar{x})) \, dx^1 dx^2 dx^3,$$

where $x^0 = \{\tau + x_1^2 + x_2^2 + x_3^2\}^{\frac{1}{2}} \geq 0$. In terms of A , we can write this as

$$Mg(\tau) = \int_{a(\tau)}^{\infty} h(\sigma + \tau, \tau) \sigma^{\frac{1}{2}} (\sigma + \tau)^{-\frac{1}{2}} d\sigma,$$

where $h = Ag$ and $a(\tau) = \max(0, -\tau)$, *i.e.* the region of integration is the intersection of the intervals $\sigma \geq 0$ and $\sigma + \tau \geq 0$. We know that Mg vanishes for large enough arguments and that it is infinitely differentiable except at the origin.

Let us denote by v^k , ($k = \infty$ permitted), a function which is k times continuously differentiable in a neighborhood of the origin. That $Mg \in H$ is a consequence of the following three formulas, whose verification is left to the reader,

$$\begin{aligned} Mg(\tau) &= \int_{a(\tau)}^1 h(\sigma + \tau, \sigma) \sigma^{\frac{1}{2}} (\sigma + \tau)^{-\frac{1}{2}} d\sigma + v^{\infty}, \\ \int_{a(\tau)}^1 h(\sigma + \tau, \sigma) \sigma^{\mu + \frac{1}{2}} (\sigma + \tau)^{\nu - \frac{1}{2}} d\sigma &= v^{\mu + \nu}, \\ \int_{a(\tau)}^1 \sigma^{\mu + \frac{1}{2}} (\sigma + \tau)^{\nu - \frac{1}{2}} d\sigma &= c_{\mu+1, \nu} \tau^{\mu + \nu + 1} \log |\tau|^{-1} + v^{\infty}, \quad c_{\mu+1, \nu} \neq 0. \end{aligned}$$

An easy consequence of these formulas is that

$$\langle l_k, h \rangle = \sum c_{\mu+1, \nu} h_{\mu+1, \nu}(0, 0), \quad \mu + \nu = k,$$

where $h_{j,k}(\sigma, \varrho) = D_{\sigma}^j D_{\varrho}^k h(\sigma, \varrho)$. Hence, since there exists an $h \in \mathcal{D}(R^+ \times R^+)$ with arbitrary derivatives at the origin, we conclude that there exists a $g_0 \in \mathcal{D}$ such that Mg_0 has a given singular part L . Hence if $h \in H$ has this singular part, $Mg_0 - h$ has no singular part and consequently it has the form Mg_1 , $g_1 \in \mathcal{D}$. Hence $M(g_0 + g_1) = h$ so that $M\mathcal{D} = H$. The proof that M is continuous is left to the reader.

Define distributions $M^*l_k \in \mathcal{D}^*$ by putting

$$\langle M^*l_k, g \rangle = \langle l_k, Mg \rangle.$$

LEMMA 8'5. — Every $f \in \mathcal{L}^+$ with support at the origin is a finite linear combination of the M^*l_k , ($k=1, 2, \dots$).

Proof. — Let B_n be all distributions in question whose orders are $\leq 2n$. We know from Lemma 8'2 that $\dim B_n = n+1$. Some reflection shows that $M^*l_1, \dots, M^*l_{n+1}$ are all in B_n and since they are linearly independent, they span B_n .

Exercise. — Show that $M^*l_1 = \frac{1}{2}\pi\delta$.

We can now prove our first main result.

THEOREM 8'1. — The mapping M^* is a linear homeomorphism

$$H^* \rightarrow \mathcal{L}^+.$$

Proof. Clear that $M^*H^* \subseteq \mathcal{L}^+$. We shall see that the equality sign holds. Let $f \in \mathcal{L}^+$. By Lemma 8'1 we know that there is an $F_0 \in \mathcal{D}(R)^*$ such that

$$\langle f, g \rangle = \langle F_0, Mg \rangle, \quad g \in \mathcal{D}(Q).$$

But F has an extension F to H^* , e.g.

$$\langle F_1, h \rangle = \langle F_0, h - \log|\tau|^{-1}\varphi L_n h \rangle,$$

where n exceeds the order of F_0 in $|\tau| \leq 1$, and the function φ is as in Lemma 8'3. But then $f - M^*F_1$ vanishes outside the origin so that by Lemma 8'5, it has the form $\sum c_k M^*l_k$. Hence $f = M^*(F_1 + \sum c_k l_k)$. We leave out the topological part, which follows from known abstract theorems.

Theorem 8'1 has the following corollary.

LEMMA 8'6. — The space H is reflexive.

Proof. — Let $G \in H^{**}$. Then $G(F)$, $F \in H^*$, is a continuous linear function of $M^*F \in \mathcal{L}^+ \subset \mathcal{D}^*$. Extending it to \mathcal{D}^* , we find, since $\mathcal{D}^{**} = \mathcal{D}$, that there is a g in \mathcal{D} such that

$$G(F) = \langle M^*F, g \rangle = \langle F, Mg \rangle.$$

Hence $G(F) = \langle F, h \rangle$ where $h \in H$. This shows that H is semireflexive. We leave out the proof that H and H^{**} have the same topologies.

Exercise. — Show that $M^*P\tau^{-1}$ and 1 are a basis for all solutions $f \in \mathcal{L}^+$ of $\square f = 0$ and show that $\square M^*j_0 = -2\pi\delta$. In a more familiar notation one has

$$M^*j_0 = \frac{1}{4}\delta(xx), \quad M^*P\tau^{-1} = \frac{1}{4}P(xx)^{-1}.$$

Let us now describe \mathcal{L}^- . Let C be the cone $xx \geq 0$. Our first observation is that any $f \in \mathcal{L}^-$ vanishes outside C . In fact, let \mathcal{O} be a bounded open set in the region $xx \leq 0$ which is symmetric around the plane $x_0 = 0$. We know that there is an $F \in \mathcal{D}(I(\mathcal{O}))^*$ such that $\langle f, g \rangle = \langle F, Mg \rangle$, $g \in \mathcal{D}(\mathcal{O})$. A reflection in the plane $x_0 = 0$ does not change the right side, but it reverses the left side. Hence $f = 0$ outside C . But then we can and will identify \mathcal{L}^- with a subset of $\mathcal{D}(C)^*$. Let

$$(M_1 g)(\tau) = \int g(x) \delta(\tau - xx) \operatorname{sgn} x_0 dx$$

be the antisymmetric mean value. It is easy to see that M_1 is continuous from $\mathcal{D}(C)$ onto $H_1 = \mathcal{D}(R^+)$ and we have

THEOREM 8'2. - The adjoint M_1^* is a linear homeomorphism

$$H_1^* \rightarrow \mathcal{L}^-.$$

The space H_1 is reflexive.

Exercise. - Let $p \neq 0$ be a polynomial. Show that the mapping $H_1 \ni h \rightarrow \rightarrow ph \in H_1$ has a bounded inverse and deduce from this that $pH_1^* = H_1^*$.

Exercise. - Define distributions

$$T_\tau, \quad T_{1\tau}, \quad A_k = M^* j_k, \quad B_k = M^* l_k \quad \text{and} \quad A_{1k} = M_1^* j_k$$

by

$$\langle T_\tau, g \rangle = Mg(\tau), \quad \langle T_{1\tau}, g \rangle = M_1 g(\tau), \quad \tau \neq 0$$

$$Mg(\tau) \sim \sum \tau^k (\log |\tau|^{-1} \langle B_k, g \rangle + \langle A_k, g \rangle),$$

$$M_1 g(\tau) \sim \sum \tau^k \langle A_{1k}, g \rangle, \quad \tau \text{ small.}$$

Let \mathcal{L}_τ^\pm be all distributions in \mathcal{L}^\pm with supports in $xx = \tau$. Let $k = 0, 1, 2, \dots$ and put $D = d/d\tau$. Show that

$$(1) \quad \mathcal{L}_\tau^+ \quad \text{has the basis } D^k T_\tau, \quad \tau \neq 0$$

$$(2) \quad \mathcal{L}_0^+ \quad \text{has the basis } A_k, B_{k+1}$$

$$(3) \quad \mathcal{L}_\tau^- \quad \text{has the basis } D^k T_{1\tau} \quad \text{when } \tau > 0$$

$$(4) \quad \mathcal{L}_0^- \quad \text{has the basis } A_{1k}.$$

Let \mathcal{L}_τ , ($\tau \geq 0$) be all invariant distributions with supports in $xx = \tau$, $x^0 \geq 0$. Show that $D^k(T_\tau + T_{1\tau})$ is a basis for \mathcal{L}_τ , ($\tau > 0$), and that $A_k + A_{1k}$, B_{k+1} is a basis for \mathcal{L}_0 .

Exercise. — Show that A_{10} is a basis for all solutions $f \in \mathcal{L}^-$ of $\square f = 0$. (Show first that j_0 is a basis for all $F \in H_1^*$ with $I^*F = 0$.) In a more familiar notation

$$A_{10} = \delta(x) \operatorname{sgn} x_0.$$

9. — Fourier transforms of Lorentz invariant distributions (L. GÄRDING).

With trivial modifications, the preceding section remains true if we change the space \mathcal{D} to \mathcal{S} throughout. This will change \mathcal{L} , \mathcal{L}^\pm , $H = M\mathcal{D}$ and $H_1 = M_1\mathcal{D}$ to $\mathcal{L} \cap \mathcal{S}^*$, $\mathcal{L}^\pm \cap \mathcal{S}^*$, $M\mathcal{S}$ and $M_1\mathcal{S}$ respectively. In this Section, however, we shall keep the old notations with the understanding that they should refer to \mathcal{S} instead of to \mathcal{D} .

Let A be a Lorentz transformation and let $f \in \mathcal{S}^*$, $g \in \mathcal{S}$. Then $A\mathcal{F}g = \mathcal{F}Ag$ and $A\mathcal{F}f = \mathcal{F}Af$ by direct computation, so that

$$\mathcal{F}A = A\mathcal{F}$$

when both factors operate on \mathcal{S}^* . It follows that

$$\mathcal{F}\mathcal{L} = \mathcal{L}, \quad \mathcal{F}\mathcal{L}^\pm = \mathcal{L}^\pm,$$

since \mathcal{F} has a bounded inverse. Let us denote the restrictions of \mathcal{F} to \mathcal{L}^\pm transported to H and H_1 by E^* and E_1^* respectively so that

$$E^* = M^{*-1}\mathcal{F}M^*, \quad E_1^* = M_1^{*-1}\mathcal{F}M_1^*.$$

By virtue of the Theorems 8'1 and 8'2, they are linear homeomorphisms $H^* \rightarrow H^*$ and $H_1^* \rightarrow H_1^*$ respectively. Let us compute their adjoints E and E_1 .

LEMMA 9'1. — One has

$$E = \text{all pairs } \{Mg, M\mathcal{F}g\},$$

$$E_1 = \text{all pairs } \{M_1g, M_1\mathcal{F}g\},$$

where $g \in \mathcal{S}$, and E and E_1 are linear homeomorphisms of H and H_1 respectively such that $E^2 = 1$ and $E_1^2 = -1$.

Proof. — We know that $H^{**} = H$. Let Mg be an element of H and put $h = EMg$ and let $F \in H^*$. Then

$$\begin{aligned} \langle F, h \rangle &= \langle F, EMg \rangle = \langle E^*F, Mg \rangle = \\ &= \langle M^{*-1}\mathcal{F}M^*F, Mg \rangle = \langle \mathcal{F}M^*F, g \rangle = \langle M^*F, \mathcal{F}g \rangle = \langle F, M\mathcal{F}g \rangle, \end{aligned}$$

so that $h = M\mathcal{F}g$. Since $M\mathcal{F}\mathcal{F}g = Mg$, $E^{-1} = E$. The arguments for E_1 are analogous. Hence the Lemma follows. We have, formally,

$$\begin{aligned} M\mathcal{F}g(\tau) &= \int \delta(\tau - xx) dx \int \exp[-2\pi i x \xi] g(\xi) d\xi = \\ &= \int \delta(\tau - xx) \exp[-2\pi i x \eta] dx d\sigma \int \delta(\sigma - \xi\xi) g(\xi) d\xi = \int \Delta^{(1)}(\tau, \sigma) Mg(\sigma) d\sigma, \end{aligned}$$

where

$$\Delta^{(1)}(\tau, \sigma) = \int \delta(\tau - xx) \exp[-2\pi i x \eta] dx, \quad \eta\eta = \sigma.$$

Hence E has the « kernel » $\Delta^{(1)}$ and we find that E_1 has the « kernel »

$$\Delta(\tau, \sigma) = \int \delta(\tau - xx) \exp[-2\pi i x \eta] \operatorname{sgn} x_0 dx, \quad \eta\eta = \sigma.$$

Exercise. — We know that $\mathcal{F}^{-1}\square\mathcal{F}$ is multiplication by $-4\pi^2xx$. Use this to show that $E^{-1}IE$ and $E^*I^*E^{*-1}$ are both multiplications by $-\pi^2\tau$ on H and H^* respectively. Prove the same result for H_1 .

Exercise. — Let $p \neq 0$ be a polynomial. Show that $p(\square)\mathcal{E}^\pm = \mathcal{E}^\pm$. (It suffices to show that $p(I^*)H^* = H^*$ and $p(I^*)H_1^* = H_1^*$: Use the preceding exercise and the fact that $p(\tau)H^* = H^*$ and $p(\tau)H_1^* = H_1^*$.)

Remark. — This result is true also relative to \mathcal{D} .

LITERATURE

P. METHÉE: *Comm. Math. Helv.*, **28**, 225 (1954); **32**, 153 (1957); *C. R. Paris* **240**, 1179 (1955).

Méthée's theory deals with Lorentz invariant distributions in any number of variables. The simplified version given here is due to J. E. ROOS and L. GÅRDING.

10. — Laplace transform (J. L. LIONS).

Preliminary remark: it seems that nobody uses $\int \exp[-2\pi px]f(x)dx$ as definition of the Laplace transform. In order to agree with common use, we shall set in this chapter:

$$\mathcal{F}f(x) = \hat{f}(x) = \int \exp[-ixy]f(y)dy.$$

Then, the Fourier inversion formula gives:

$$f(x) = \overline{\mathcal{F}}\hat{f} = \left(\frac{1}{2\pi}\right)^n \int \exp[ixy]\hat{f}(y)dy.$$

10'1. General remarks. — In Fourier transform theory, we consider a distribution T which belongs to \mathcal{S}^* ; obviously, one can also consider distributions T such that $\exp[-\xi x]T \in \mathcal{S}^*$ for some $\xi \in R^n$. But assume now that T has the property: $\exp[-\xi x]T \in \mathcal{S}^*$, $\exp[-\xi' x]T \in \mathcal{S}^*$, for $\xi \neq \xi'$. This implies that $\exp[-\xi'' x]T \in \mathcal{S}^*$ for every ξ'' on the segment joining ξ to ξ' . In a general way we have:

THEOREM 10'1. — Let $T \in \mathcal{D}^*$; we denote by Γ_T the set (possibly void) of all the $\xi \in R^n$ such that $\exp[-\xi x]T \in \mathcal{S}^*$. This set is *convex*.

Proof. — Let ξ^0 and ξ^1 be in Γ_T . Consider $\xi = t\xi^0 + (1-t)\xi^1$, $0 < t < 1$. We set: $a(x, \xi) = \exp[-x\xi]/(\exp[-x\xi^0] + \exp[-x\xi^1])$. This function is bounded with all of its derivatives. In particular: $a \in \mathcal{O}_M$. Now

$$\exp[-x\xi]T = a(\exp[-x\xi^0]T) + a(\exp[-x\xi^1]T),$$

and since $a \in \mathcal{O}_M$, this belongs to \mathcal{S}^* .

We denote by Γ_T^0 the interior of Γ_T . We have

THEOREM 10'2. — Same notation than in Theorem 10'1. For every $\xi \in \Gamma_T^0$, one has: $\exp[-x\xi]T \in \mathcal{O}'_c$; and moreover, $\exp[-x\xi]T$ remain in a bounded set of \mathcal{O}'_c when $\xi \in K =$ compact set of Γ_T^0 .

Proof. — For $\varepsilon > 0$ small enough, we can find $\xi^1, \xi^2, \dots, \xi^M \in \Gamma_T^0$ such that the set: $\xi + b$, $\xi \in K$, $|b| \leq \varepsilon$, is contained in the convex hull of ξ^1, \dots, ξ^M .

We set:

$$a(x, \xi) = \exp[-x\xi]/(\exp[-x\xi^1] + \dots + \exp[-x\xi^M]).$$

It is easily checked that $a \in \mathcal{S}$; if we write

$$\exp[-x\xi]T = \sum_{i=1}^{i=M} a(\exp[-x\xi^i]T),$$

we can conclude that $\exp[-x\xi]T \in \mathcal{O}'_c$ (Exercise in 7b). Since a remains in a bounded set of \mathcal{S} (when $\xi \in K$), one can verify that $\exp[-x\xi]T$ remains in a bounded set of \mathcal{O}'_c .

Moreover since the function

$$\xi \rightarrow a(\cdot, \xi)$$

is infinitely differentiable from K into \mathcal{S} , one has

THEOREM 10'3. — The function $\xi \rightarrow \exp[-x\xi]T$ is infinitely differentiable from Γ_T^0 into \mathcal{O}'_c .

Definition of the Laplace Transform. — For T given in \mathcal{D}^* , we set

$$\mathcal{L}(T) = \mathcal{F}(\exp[-x\xi]T),$$

define for $\xi \in \Gamma_T$. This is the *Laplace transform* of T . If Γ_T is void, one says that T does not admit a Laplace transform. If Γ_T is a point, we get essentially the Fourier transform. The most interesting case is when Γ_T^0 is non void. We study this case in 10'2. By Theorems 10'2 and 10'3 we know already:

Proposition 10'1 — $\mathcal{L}(T) = G(\xi, \eta)$; $G(\xi, \cdot) \in \mathcal{O}_M$ for $\xi \in \Gamma_T^0$, and $\xi \rightarrow G(\xi, \cdot)$ is infinitely differentiable from Γ_T^0 into \mathcal{O}_M .

Exercise. — To calculate the Laplace transform of $\delta, \frac{\partial}{\partial x_i}(\delta)$.

Exercise. — If $\mathcal{C} \in \mathcal{C}^*$, $\Gamma_{\mathcal{C}} = R^n$; what is Γ_T if $T(x) = \exp[-x^2]$?

10'2. Space $\mathcal{S}^*(\Gamma)$. Laplace transform in $\mathcal{S}^*(\Gamma)$. — We give $\Gamma =$ convex open set in R^n . By $\mathcal{S}^*(\Gamma)$ we mean the space of distributions T such that: $\exp[-x\xi]T \in \mathcal{S}^*$ for every $\xi \in \Gamma$. We define a « topology » into $\mathcal{S}^*(\Gamma)$ in this way: $T_n \rightarrow 0$ into $\mathcal{S}^*(\Gamma)$ if $\exp[-x\xi]T_n \rightarrow 0$ into \mathcal{S}^* for every $\xi \in \Gamma$, uniformly for $\xi \in$ compact set of K .

By 10'1 we know that $\exp[-x\xi]T \in \mathcal{O}'_c$ for every $\xi \in \Gamma$. One can prove that $T_n \rightarrow 0$ into $\mathcal{S}^*(\Gamma)$ is equivalent with: $\exp[-x\xi]T_n \rightarrow 0$ in \mathcal{O}'_c for every $\xi \in \Gamma$.

For $T \in \mathcal{S}^*(\Gamma)$, we set

$$(2.1) \quad \mathcal{L}(T) = \mathcal{F}(\exp[-x\xi]T).$$

We are going to prove the first main result:

THEOREM 10'4. — For $T \in \mathcal{S}^*(\Gamma)$, $\mathcal{L}(T)$ is an holomorphic function of $(\xi_j + i\eta_j) = (p_j) = p$, in the tube $\Gamma + iR^n$ (i.e. $p = \xi + i\eta$, $\xi \in \Gamma$, $\eta \in R^n$). Let

$$(2.2) \quad \mathcal{L}(T) = F(p)$$

be this function. The function $\eta \rightarrow F(\xi + i\eta)$ belongs to a bounded set in \mathcal{O}_M when $\xi \in$ compact set of Γ .

Proof. — It remains only to prove that $F(p)$ is holomorphic, i.e. that

$$(2.3) \quad \left(\frac{\partial}{\partial \xi_j} + i \frac{\partial}{\partial \eta_j} \right) \mathcal{L}(T) = 0.$$

But:

$$\begin{aligned}\frac{\partial}{\partial \xi_j} \mathcal{L}(T) &= \mathcal{F} \left(\frac{\partial}{\partial \xi_j} (\exp[-x\xi]T) \right) = -\mathcal{F}(x_j \exp[-x\xi]T) = \\ &= -i \frac{\partial}{\partial \eta_j} (\exp[-x\xi]T) . \quad q.e.d.\end{aligned}$$

The second fundamental result is the reciprocal of Theorem 10'4:

THEOREM 10'5. — Let $F(p)$ be an holomorphic function in the tube $\Gamma + iR^n$, with the property: $\eta \rightarrow F(\xi + i\eta) \in \mathcal{O}_M$ for every $\xi \in \Gamma$, and moreover: $F(\xi + i\eta)$ belongs to a bounded set in \mathcal{D}_M when $\xi \in$ compact set of Γ . Then in a unique way:

$$F(p) = \mathcal{L}(T), \quad T \in \mathcal{S}^*(\Gamma).$$

More precisely:

$$\exp[-x\xi]T = \overline{\mathcal{F}}_\eta(F(\xi + i\eta)),$$

i.e.,

$$T(x) = \frac{1}{(2\pi)^n} \int \exp[xp] F(\xi + i\eta) d\eta.$$

(*Caution:* this last equation has to be taken in the distributions sense!).

Proof. — We consider: $\overline{\mathcal{F}}_\eta(F(\xi + i\eta)) = S^{(\xi)} \in \mathcal{O}'_c$. The function $\xi \rightarrow F(\xi + i\eta)$ is infinitely differentiable from Γ into \mathcal{O}_M : Then $\xi \rightarrow S^{(\xi)}$ is infinitely differentiable from Γ into \mathcal{O}'_c : We have:

$$\frac{\partial}{\partial \xi_j} S^{(\xi)} = \overline{\mathcal{F}}_\eta \left(\frac{\partial}{\partial \xi_j} F \right) = \overline{\mathcal{F}}_\eta \left(-i \frac{\partial}{\partial \eta_j} F \right) = -x_j S^{(\xi)}.$$

If we set:

$$U^{(\xi)} = \exp[x\xi] S^{(\xi)}, \quad \text{then} \quad \frac{\partial}{\partial \xi_j} U^{(\xi)} = 0,$$

i.e. $U^{(\xi)}$ does not depend on ξ : $U^{(\xi)} = T$. Conclusion:

$$\overline{\mathcal{F}}_\eta(F(\xi + i\eta)) = \exp[-x\xi]T;$$

Theorem 10'5 follows readily.

Remark. — Assume that $F(\xi + i\eta)$ is holomorphic in $\Gamma + iR^n$ and that $|F(\xi + i\eta)| \leq \text{polynomial in } |\eta|$ when $\xi \in$ compact set K in Γ . Then, it follows, by using the Cauchy integral, that $F(\xi + i\eta) \in$ bounded set of (\mathcal{O}_M) when $\xi \in K$.

10'3. *Properties of the Laplace transform.* — We assume again that Γ is a convex open set of R^n .

THEOREM 10'6. — If $S, T \in \mathcal{S}^*(\Gamma)$, then $S * T \in \mathcal{S}^*(\Gamma)$. In other words $\mathcal{S}^*(\Gamma)$ is a convolution algebra.

Proof. — As it is easily checked,

$$(3.1) \quad (\exp[-x\xi]S) * (\exp[-x\xi]T) = \exp[-x\xi](S * T);$$

we know that \mathcal{O}'_0 is a convolution algebra, hence the result.

Exercise. — If $S, T \in \mathcal{S}^*(\Gamma)$, $S * T = 0$, then either S or T is zero.

THEOREM 10'7. — If $S, T \in \mathcal{S}^*(\Gamma)$, $\mathcal{L}(S * T) = (\mathcal{L}S)(\mathcal{L}T)$.

Proof. — From (3.1).

Exercise. — Let $P = P(\partial/(x))$ be a differential operator with constant coefficients. Then $P\delta \in \mathcal{E}^*$, and in particular $P\delta \in \mathcal{S}^*(\Gamma)$; $\mathcal{L}(PT) = P(p)(\mathcal{L}T)$.

Exercise:

a) Let $T \in \mathcal{S}^*(\Gamma)$ and assume that $T = 0$ when $x\xi < A$. Then, if $\xi^0 \in \Gamma$, one has $\xi^0 + t\xi \in \Gamma$ for every $t \geq 0$;

b) $\exp[tB] \exp[-x(\xi^0 + t\xi)]T$ remains in a bounded set of \mathcal{S}^* when $t \geq 0$;

c) Reciprocally, consider an open convex set Γ such that $\xi^0 \in \Gamma$ implies: $\xi^0 + t\xi \in \Gamma$ for every $t \geq 0$. Let T be in $\mathcal{S}^*(\Gamma)$ with the property:

$\exp[tB] \exp[-x(\xi^0 + t\xi)]T \in$ bounded set of \mathcal{S}^* when $t \geq 0$, for every $B < A$. Then $T = 0$, if $x\xi < A$. Translate this in terms of $F(p) = \mathcal{L}(T)$.

Exercise. — Consider $T \in \mathcal{D}^*_+$ in R^1 . Then Γ_T is either void or a half plane: $\xi > \xi_0$.

LITERATURE

SCHWARTZ: *Transformation de Laplace des distributions; Sem. Math. Lund.*, (1952), p. 196.

LIONS: *Support dans la transformation de Laplace. Journ. d'Anal. Israel*, t. II, (1952-53), p. 369.

11. — Vectorial distributions (J. L. LIONS).

11'1. — Let E be a linear topological space. Let \mathcal{O} be an open set in R^n . We want to define distributions in \mathcal{O} , with values in E . Of course if E is one dimensional, say $E = C$, the distributions with values in E have to coincide with elements of $\mathcal{D}^*(\mathcal{O})$. But, by definition, $\mathcal{D}^*(\mathcal{O}) = \mathcal{L}(\mathcal{D}(\mathcal{O}); E)$ space of continuous linear mappings from $\mathcal{D}(\mathcal{O})$ into $E = C$.

This leads to the following general definition: the space $\mathcal{D}^*(\mathcal{O}; E)$ of distributions on \mathcal{O} with values in E is: $\mathcal{D}^*(\mathcal{O}; E) = \mathcal{L}(\mathcal{D}(\mathcal{O}); E)$, space of continuous linear mappings from $\mathcal{D}(\mathcal{O})$ into E .

This is algebraic. We define now a topology onto $\mathcal{D}(\mathcal{O}; E)$, as follows: let p be a seminorm on E , and B be a bounded set in $\mathcal{D}(\mathcal{O})$; if u belongs to $\mathcal{D}^*(\mathcal{O}; E)$, we set

$$q_{p,B}(u) = \sup p(u(f)), \quad f \in B.$$

We define in this way a seminorm on $\mathcal{D}^*(\mathcal{O}; E)$, and when p and B vary, the $q_{p,B}$ define the topology on $\mathcal{D}^*(\mathcal{O}; E)$. (In other words: let V be a neighborhood of 0 in E ; the set of all u such that $u(f) \in V$ for $f \in B$ defines a neighborhood of 0 in $\mathcal{D}^*(\mathcal{O}; E)$).

Remark. – This is the general way to define a topology on $\mathcal{L}(\mathcal{A}; \mathcal{B})$ = space of continuous linear mappings from \mathcal{A} to \mathcal{B} , \mathcal{A} and \mathcal{B} being two linear topological spaces. Example: $\mathcal{L}(\mathcal{S}; \mathcal{S})$; this defines the topology on \mathcal{O}_M and \mathcal{O}'_c .

Exercise. – To check that when $E = C$, $\mathcal{L}(\mathcal{D}(\mathcal{O}); E)$ has the same topology that $\mathcal{D}^*(\mathcal{O})$.

We admit here the following rather technical Lemma:

LEMMA 11'1. – If u is a linear mapping from $\mathcal{D}(\mathcal{O})$ into E which transforms bounded sets in $\mathcal{D}(\mathcal{O})$ into bounded sets in E , then u is *continuous*.

This Lemma has an important consequence:

THEOREM 11'1. – If E is complete, then $\mathcal{D}^*(\mathcal{O}; E)$ is complete.

Proof. – Let u_n be a Cauchy sequence in $\mathcal{D}^*(\mathcal{O}; E)$. This means that $q_{p,B}(u_n)$ is a Cauchy sequence in C . In particular, for every $f \in \mathcal{D}(\mathcal{O})$ and every p , $p(u_n(f))$ is a Cauchy sequence. Since we assume that E is complete, it follows that $u_n(f) \rightarrow u(f)$ in E . This defines a linear mapping: $f \rightarrow u(f)$ from $\mathcal{D}(\mathcal{O})$ into E . The main part of the proof consists in proving that this mapping is *continuous*. Take B , bounded set in $\mathcal{D}(\mathcal{O})$, and let p be an arbitrary seminorm on E . Since $q_{p,B}(u_n)$ is a Cauchy sequence, $p(u(f))$ is bounded when $f \in B$. This means that u transforms B into a bounded set in E , and by Lemma 11'1, u is continuous. The « proof » is completed (except that we consider sequences instead of filters).

Example. – $\mathcal{D}^*(\mathcal{O})$ is complete.

Other example: we define the space of tempered distributions with values in E , say $\mathcal{S}^*(E)$, by

$$\mathcal{S}^*(E) = \mathcal{L}(\mathcal{S}; E).$$

An analogous Lemma to Lemma 11'1 holds; then, if E is complete, $\mathcal{S}^*(E)$ is complete.

We always assume in the following that E is complete.

Remark. 11'1. — A special case of fundamental importance is the case when $E = \mathcal{D}^*(\mathcal{O}_1)$, $\mathcal{O}_1 =$ open set in R^m . We shall study this case in Section 12.

11'2. *The technical theory of vectorial distributions.*

a) Consider $T \in \mathcal{D}^*(\mathcal{O})$, $e \in E$; then the mapping

$$f \rightarrow \langle T, f \rangle e = \left(\int T(x) f(x) dx \right) e,$$

is a continuous linear mapping from $\mathcal{D}(\mathcal{O})$ into E . It defines an element of $\mathcal{D}^*(\mathcal{O}; E)$, denoted by $T \otimes e$.

Functional notation for vectorial distributions: if $T \in \mathcal{D}^*(\mathcal{O}; E)$ we shall write also $T(x)$, so that $T(f) = \int T(x) f(x) dx$: the situation is the same as in the ordinary case ($E = C$), except that here the integral is taken in E . Note that this is formal.

Consider now a function $F(x)$, continuous from \mathcal{O} to E (this means: for every semi-norm p , $p(F(x))$ is an ordinary continuous function on \mathcal{O}). If $f \in \mathcal{D}(\mathcal{O})$ one can consider the integral $\int F(x) f(x) dx$, which defines an element of E and a continuous linear mapping $f \rightarrow \int F(x) f(x) dx$ from $\mathcal{D}(\mathcal{O})$ to E (this is a theorem!). Then $F(x)$ defines a distribution $\in \mathcal{D}^*(\mathcal{O}; E)$, always denoted by $F(x)$.

b) Supports. Straightforward.

Exercise. — $\text{supp } (T \otimes e) = \text{supp } (T)$.

c) Derivatives.

$$D^\alpha T(f) = (-1)^{|\alpha|} T(D^\alpha f), \quad T \in \mathcal{D}^*(\mathcal{O}; E), \quad f \in \mathcal{D}(\mathcal{O}).$$

D^α is a continuous linear mapping of $\mathcal{D}^*(\mathcal{O}; E)$ into itself.

Exercise. — $\text{supp } (D^\alpha T) \subset \text{supp } (T)$.

Exercise. — $D^p(T \otimes e) = (D^p T) \otimes e$.

d) *Multiplication*: let a be in $\mathcal{E}(\mathcal{O})$, $T \in \mathcal{D}^*(\mathcal{O}; E)$. We define aT by

$$aT(f) = T(af), \quad f \in \mathcal{D}(\mathcal{O}),$$

and this defines a continuous linear mapping $T \rightarrow aT$ of $\mathcal{D}^*(\mathcal{O}; E)$ into itself.

Exercise. — $a(T \otimes e) = (aT) \otimes e$.

e) *Regularization*. - Assume to simplify that $\mathcal{O} = R^n$. Let φ be in $\mathcal{D} = \mathcal{D}(R^n)$; we define

$$T * \varphi(x) = \int T(y) \varphi(x - y) dy.$$

One proves easily that $x \rightarrow T * \varphi(x)$ is an infinitely differentiable function in R^n with values in E . If $\varphi_\varepsilon \in \mathcal{D} \rightarrow \delta$ in \mathcal{E}^* , $T * \varphi_\varepsilon \rightarrow T$.

Exercise. - $(T \otimes e) * \varphi = (T * \varphi) \otimes e$, $T \in \mathcal{D}^*(\mathcal{O})$, $e \in E$.

Exercise. - $D^p(T * \varphi) = (D^p T) * \varphi$, $T \in \mathcal{D}^*(\mathcal{O}; E)$, $\varphi \in \mathcal{D}(\mathcal{O})$.

f) *Local structure of vectorial distributions*. - Let \mathcal{O}_1 be a bounded open set, $\mathcal{O}_1 \subset \mathcal{O}_1 \subset \mathcal{O}$. Every $T \in \mathcal{D}^*(\mathcal{O}; E)$ defines in an obvious way an element of $\mathcal{D}^*(\mathcal{O}_1; E)$ (the restriction of T to \mathcal{O}_1). We shall say that T is of finite order in \mathcal{O}_1 , say m , if the mapping

$$f \rightarrow T(f)$$

of $\mathcal{D}(\mathcal{O}_1)$ into E is continuous for $\mathcal{D}(\mathcal{O}_1)$ provided with the topology induced by $\mathcal{D}^m(\mathcal{O}_1)$.

If $E = C$ we know that every distribution is of finite order on \mathcal{O}_1 .

Caution: this can be false in an infinitely dimensional space. We give the following very simple counter example: take $E = \mathcal{D}(\mathcal{O})$, and $T =$ identity mapping. Then T restricted to \mathcal{O}_1 is simply the mapping $f \rightarrow f$ from $\mathcal{D}(\mathcal{O}_1)$ into $\mathcal{D}(\mathcal{O}_1)$, and this mapping is never continuous on $\mathcal{D}(\mathcal{O}_1)$ provided with the topology of $\mathcal{D}^{(m)}(\mathcal{O}_1)$.

A positive result is:

THEOREM 11'2. - If E is a Banach space, every $T \in \mathcal{D}^*(\mathcal{O}; E)$ is of finite order on \mathcal{O}_1 . We admit this result.

11'3. Fourier transform. - We have already defined $\mathcal{S}^*(E) = \mathcal{L}(\mathcal{S}; E)$. If $T \in \mathcal{S}(E)$, $f \in \mathcal{S}$, we set

$$(3.1) \quad \mathcal{F}T(f) = T(\mathcal{F}f).$$

The mapping $f \rightarrow T(\mathcal{F}f)$ is continuous of \mathcal{S} to E ; then (3.1) defines $T \in \mathcal{S}^*(E)$ and

THEOREM 11'3. - \mathcal{F} is an isomorphism of $\mathcal{S}^*(E)$ onto itself, with inverse $\bar{\mathcal{F}}$.

Exercise. — If $E = C$ the above definition of \mathcal{FT} coincides with the previous one.

Exercise. — $\mathcal{F}(D^p T) = (2\pi i x)^p \mathcal{F}T$.

Exercise (Laplace transform):

a) Let I be an open set convex in R^n ; by $\mathcal{S}_I^*(E)$ we mean the space of distributions $T \in \mathcal{D}^*(R^n; E)$ such that $\exp[-x\xi]T \in \mathcal{S}^*(E)$ for every $\xi \in I$. Define the topology on $\mathcal{S}_I^*(E)$. Prove that $\mathcal{S}_I^*(E) = \mathcal{S}^*(I)$, if $E = C$.

b) We define $\mathcal{L}(T) = \mathcal{F}(\exp[-x\xi]T)$. What kind of element is this? Consider $T = S \otimes e$, $S \in \mathcal{S}^*(I)$, $e \in E$. Prove that $\mathcal{L}(D^\alpha T) = p^\alpha \mathcal{L}(T)$.

11.4. *An example of convolutions of distributions with vectorial values.* — Let E, F, G be three Banach spaces; then $\mathcal{L}(E; F)$, $\mathcal{L}(F; G)$, and so on are Banach spaces. To simplify the writing, we set

$$\mathcal{E}^*(R^n; E) = \mathcal{E}^*(E), \quad \mathcal{D}^*(R^n; E) = \mathcal{D}^*(E).$$

We consider now $S \in \mathcal{D}^*(\mathcal{L}(E; F))$, $T \in \mathcal{E}^*(\mathcal{L}(F; G))$; then one can define:

$$(4.1) \quad \int T(x-y)S(y)dy = T * S(x), \quad T * S \in \mathcal{D}^*(\mathcal{L}(E; G))$$

(If S and T are continuous functions with values in $\mathcal{L}(E; F)$ and $\mathcal{L}(F; G)$ formula (4.1) is easily understood: for every x , $T(x-y)S(y) \in \mathcal{L}(E; G)$, and depends continuously on y . Moreover, T being with compact support, the integral is extended to a compact set in R^n , so that the integral converges in $\mathcal{L}(E; G)$. You can think of the general case by passing to the limit: if $T_n \rightarrow T$, $S_n \rightarrow S$ in $\mathcal{E}^*(\mathcal{L}(F; G))$, $\mathcal{D}^*(\mathcal{L}(E; F))$, then one can prove that $T_n * S_n$ converges in $\mathcal{D}^*(\mathcal{L}(E; G))$, to $T * S$ by definition; details of the proof are rather involved.)

If $G = E$, one can define $S * T \in \mathcal{D}^*(\mathcal{L}(F; F))$.

LITERATURE

SCHWARTZ: *Théorie des distributions à valeurs vectorielles. Annales Institut Fourier* t. VII (1957), p. 1.

12. — The nuclear theorem of Schwartz (J. L. LIONS).

12.1. — Notations: we consider $R^n = X^n$, $x \in X^n$, $R^m = Y^m$, $y \in Y^m$, the spaces $\mathcal{D}(X^n)$, $\mathcal{D}^*(X^n)$, ..., are denoted by:

$$\mathcal{D}_{x'} \mathcal{D}_{x'}^* \mathcal{D}_y \mathcal{D}_y^*.$$

We consider also $X^n \times Y^m$; $\mathcal{D}(X^n \times Y^m) = \mathcal{D}_{xy}$, $\mathcal{D}^*(X^n \times Y^m) = \mathcal{D}_{xy}^*$.

We want to study the space

$$(1.1) \quad J = \mathcal{L}(\mathcal{D}_x; \mathcal{D}_y^*).$$

By Section 11 we know that this space is *complete*. The main theorem we want to prove is:

THEOREM 12'1. – (Nuclear Theorem of Schwartz): $\mathcal{L}(\mathcal{D}_x; \mathcal{D}_y^*) = \mathcal{D}_{xy}^*$, algebraically and topologically.

The sign $=$ has to be interpreted in a suitable way, as explained in the following. We divide the proof in three steps. (The following method is due to L. EHRENPREIS).

12'2. *The mapping defined by \mathcal{D}_{xy}^* .* – Let T be a distribution given in \mathcal{D}_{xy}^* . We can associate to T a continuous linear mapping, say u_T , of \mathcal{D}_x into \mathcal{D}_y^* in the following way: if

$$f \in \mathcal{D}_x, \quad g \rightarrow \langle T, f(x)g(y) \rangle = \iint T(x, y) f(x)g(y) dx dy,$$

is a continuous linear form on \mathcal{D}_y , then

$$(2.1) \quad \iint T(x, y) f(x)g(y) dx dy = \langle u_T(f), g \rangle, \quad u_T(f) \in \mathcal{D}_y^*.$$

It is easily checked that $f \rightarrow u_T(f)$ is continuous of \mathcal{D}_x to \mathcal{D}_y^* . In other words

$$(2.2) \quad u_T \in J.$$

LEMMA 12'1. – The mapping $T \rightarrow u_T$ of \mathcal{D}_{xy}^* into J is one to one.

Proof. – Because $\mathcal{D}_x \otimes \mathcal{D}_y$ is dense in \mathcal{D}_{xy} .

By Lemma 12'1 we can identify \mathcal{D}_{xy}^* with a subspace of J (identification of T and u_T).

In the next step we compare on \mathcal{D}_{xy}^* the natural topology and the topology induced by J .

12'3. – **LEMMA 12'2.** $\mathcal{D}_{xy}^* \subset J$ topologically; in other words: $T \rightarrow u_T$ is continuous of \mathcal{D}_{xy}^* into J .

Proof. – Assume that $T_n \rightarrow 0$ in \mathcal{D}_{xy}^* . Let B_x be a bounded set in \mathcal{D}_x . We have to prove that $u_{T_n}(f) \rightarrow 0$ in \mathcal{D}_y^* uniformly for $f \in B_x$, i.e. $u_{T_n}(f), g \rightarrow 0$ uniformly for $f \in B_x, g \in B'_y$, where B'_y is a bounded set in \mathcal{D}_y . But this equals $\langle T_n, f(x)g(y) \rangle$; when $f \in B_x, g \in B'_y$, $f(x)g(y)$ belongs to a bounded set in \mathcal{D}_{xy} , hence the result.

Actually, one can prove more, and this is the main part of the proof:

LEMMA 12'3. — The topology \mathcal{D}_{xy}^* is *identical* with the topology induced by J .

Proof. — Assume now that $u_{T_n} \rightarrow 0$ in J . Then we can conclude that, given B_x and B'_y , bounded sets in \mathcal{D}_x and \mathcal{D}_y ,

$$\langle T_n, f(x)g(y) \rangle \rightarrow 0 \quad \text{uniformly for } f \in B_x \text{ and } g \in B'_y.$$

We have to prove that this implies:

$$\langle T_n, h(x, y) \rangle \rightarrow 0 \quad \text{uniformly for } h \in B''_{xy} = \text{bounded set in } \mathcal{D}_{xy}.$$

This follows from the

LEMMA 12'4. — Let B be a bounded set in \mathcal{D}_{xy} . Then we can find B_1 and B_2 , bounded sets in \mathcal{D}_x and \mathcal{D}_y , such that, every $h \in B$ can be written

$$(3.1) \quad h(x, y) = \sum \lambda_i f_i(x) g_i(y),$$

where $\sum |\lambda_i| \leq M$, $f_i \in B_1$, $g_i \in B_2$.

Proof. — The functions of B have their supports in a fixed cube, say $] -\alpha, \alpha[^{n+m}$. We set: $a = \alpha + 1$, and $Q =] -a, +a[^{n+m}$. For $h \in B$, let h^* be the extension of h by periodicity, period a in all of the variables. Then h^* can be written as a Fourier series:

$$h^*(x, y) = \sum c_{k,l} \exp \left[\pi \frac{i}{a} kx \right] \exp \left[\pi \frac{i}{a} ly \right],$$

$$k = (k_1, \dots, k_n), \quad l = (l_1, \dots, l_m), \quad k_i, l_i \text{ integers.}$$

The sequence $c_{k,l}$ is a sequence with fast decrease, namely: for every polynomial Q , $|Q(k, l)c_{k,l}|$ is bounded by M_Q , and this uniformly when $h \in B$ (easy to verify by integrations by parts of the formulas giving the coefficients $c_{k,l}$).

We take now $\theta \in \mathcal{D}(Q)$, $= 1$ on $] -\alpha, +\alpha[^{n+m}$ $\theta(x, y) = \theta_1(x)\theta_2(y)$. We have

$$h = \theta h^* = \sum c_{k,l} \left(\theta_1 \exp \left[\frac{\pi i}{a} kx \right] \right) \left(\theta_2 \exp \left[\frac{\pi i}{a} ly \right] \right).$$

and this can be written

$$h = \sum c_{k,l} |c_{k,l}|^{-\frac{1}{2}} \left(\theta_1 |c_{k,l}|^{\frac{1}{2}} \exp \left[\frac{\pi i}{a} kx \right] \right) \left(\theta_2 |c_{k,l}|^{\frac{1}{2}} \exp \left[\frac{\pi i}{a} ly \right] \right).$$

Now:

$$\theta_1 |c_{k,l}|^{\frac{1}{2}} \exp \left[\frac{\pi i}{a} kx \right] = f_{k,l} \in \text{bounded set in } \mathcal{D}_x,$$

$$\theta_2 |c_{k,l}|^{\frac{1}{2}} \exp \left[\frac{\pi i}{a} ly \right] = g_{k,l} \in \text{bounded set in } \mathcal{D}_y;$$

and the proof is completed.

In the following step, we prove the nuclear theorem.

12'4. — LEMMA 12'5. Let $u \in \mathcal{L}(\mathcal{D}_x; \mathcal{E}_y) \subset J$. Then there exists $T \in \mathcal{D}_{xy}^*$ such that $u = u_T$.

Proof. — If $f \in \mathcal{D}_x$, $f \rightarrow u(f)(y)$ is a continuous linear form on \mathcal{D}_x , hence

$$u(f)(y) = \int S^{(y)}(x) f(x) dx, \quad S^{(y)}(x) \in \mathcal{D}_x^*.$$

The function $y \rightarrow S^{(y)}(x)$ is infinitely differentiable from Y^m into \mathcal{D}_x^* ; we set $S^{(y)}(x) = T(x, y)$. Obviously, $u = u_T$.

Exercise. — Prove an analogous result replacing \mathcal{E}_y by \mathcal{E}_y^0 .

LEMMA 12'6. — \mathcal{D}_{xy}^* is dense in J .

Proof. — Let $u \in J$. Consider ϱ_n , sequence of \mathcal{D}_y with $\varrho_n \rightarrow \delta$. For $f \in \mathcal{D}_x$ we set

$$(12'4.1) \quad u_n(f) = u(f) * \varrho_n.$$

We define in this way $u_n \in \mathcal{L}(\mathcal{D}_x; \mathcal{E}_y)$. By Lemma 12'5, $u_n = u_{T_n}$. Since $\varrho_n \rightarrow \delta$, (4.1) implies: $u_n \rightarrow u$ in S , i.e. $u_{T_n} \rightarrow u$ in J . Lemma 12'6 is proved.

Proof of Theorem 12'1. — \mathcal{D}_{xy}^* is a subspace of J , with the topology induced, Since it is complete it is closed in J . But it is dense. Conclusion: $\mathcal{D}_{xy}^* = J$.

Remark. Every $u \in \mathcal{L}(\mathcal{D}_x; \mathcal{D}^*)$ can be written in a unique way

$$u = u_T; \quad T(x, y) \text{ is the kernel of } u.$$

12'5. Some related results.

a) Let \mathcal{O} and \mathcal{O}' be open sets in R^n and R^m . The same proof as above gives:

$$\mathcal{L}(\mathcal{D}(\mathcal{O}_x); \mathcal{D}^*(\mathcal{O}'_y)) = \mathcal{D}^*(\mathcal{O}_x \times \mathcal{O}'_y).$$

b) *Exercise*: Prove that $\mathcal{E}_{xy}^* \subset \mathcal{L}(\mathcal{E}_x; \mathcal{E}_y^*)$ (by a mapping analogous to $T \rightarrow u_T$).

Actually one can prove that $\mathcal{E}_{xy}^* = \mathcal{L}(\mathcal{E}_x; \mathcal{E}_y^*)$.

c) *Exercise*: Prove that $\mathcal{S}_{xy}^* \subset \mathcal{L}(\mathcal{S}_x; \mathcal{S}_y^*)$.

One can prove that

$$\mathcal{S}_{xy}^* = \mathcal{L}(\mathcal{S}_x; \mathcal{S}_x^*).$$

12'6. *Some hints on applications.* — In many applications, we work with two (or more...) spaces of functions or distributions, say to fix the ideas:

E_x = space of distributions on \mathcal{O}_x ,

F_y = space of distributions on \mathcal{O}'_y .

Except in the case of spaces of analytic functions (but then there are much more elementary direct results), we have $\mathcal{D}(\mathcal{O}_x) \subset E_x$, $F_y \subset \mathcal{D}^*(\mathcal{O}'_y)$.

Then if u is given, continuous linear mapping of E_x into F_y , u defines in particular a continuous linear mapping of $\mathcal{D}(\mathcal{O}_x)$ into $\mathcal{D}^*(\mathcal{O}'_y)$, so that: $u = u_T$, $T \subset \mathcal{D}^*(\mathcal{O}_x \times \mathcal{O}'_y)$; of course T has supplementary properties, and it can be very difficult to write down explicitly these supplementary properties, but it is already very interesting to know that for $f \in \mathcal{D}(\mathcal{O}_x)$, $u(f) = \int_{\mathcal{O}} T(x, y) f(x) dx$.

Example. — Let D be a partial differential operator, continuous linear mapping of E into F ; E and F are spaces of functions or distributions, the boundary conditions being included in E . We prove (or try to prove) directly that D is an isomorphism of E onto F (directly means, for instance, by L^2 estimates). Then $D^{-1} = G$; the kernel of G , say $G(x, y)$, is the *Green's kernel of the problem*.

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Theory of Lie Groups.

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1. - Basic notions.

Consider a group where the elements are functions of a certain set of parameters, *i.e.* the element x is represented by the set of parameters (x^1, x^2, \dots, x^r) , which are c -numbers. Thus

$$x \equiv (x^1, x^2, \dots, x^r),$$

$$y \equiv (y^1, y^2, \dots, y^r).$$

Then the composition law $xy = z$ implies the functional dependence

$$(1) \quad z^\alpha = \varphi^\alpha(x^1, x^2, \dots, x^r; y^1, y^2, \dots, y^r) = \varphi^\alpha(x, y) \quad (\alpha = 1, \dots, r).$$

The associative law $(xy)z = x(yz)$ then implies that

$$(2) \quad \varphi^\alpha(\varphi(x, y), z) = \varphi^\alpha(x, \varphi(y, z)) \equiv 0.$$

The parameters are so chosen that the unit element I is represented by $I \equiv (0, 0, \dots, 0)$.

The idea of LIE is to consider only infinitesimal elements of the group, *i.e.* those elements which lie in the immediate neighbourhood of the unit element. This is sufficient because it is found that the properties of these characterize almost all properties of the group.

From the definition of the unit element we have that

$$(3) \quad \varphi^\alpha(x, 0) = x^\alpha, \quad \varphi^\alpha(0, y) = y^\alpha.$$

If we consider $\varphi^\alpha(x, y)$ for x and y in the neighbourhood of the unit element, so that all the x^α and y^α are infinitesimal, we can expand $\varphi^\alpha(x, y)$ in a power series in the x^α and y^α :

$$(4) \quad \varphi^\alpha(x, y) = x^\alpha + y^\alpha + a_{\beta\gamma}^\alpha x^\beta y^\gamma + g_{\beta\gamma\delta}^\alpha x^\beta x^\gamma y^\delta + h_{\beta\gamma\delta}^\alpha x^\beta y^\gamma y^\delta + 0(4),$$

(where the dummy suffix summation convention is used), for by (3) if either x or y vanishes, we have to get y^α or x^α respectively.

If we replace y by x^{-1} in (4), then the R.H.S. must vanish. Solving we find

$$(5) \quad (x^{-1})^\alpha = -x^\alpha + a_{\beta\gamma}^\alpha x^\beta x^\gamma + 0(3).$$

Further we obtain from (4) and (5):

$$(6) \quad \varphi^\alpha(x^{-1}, y^{-1}) = -x^\alpha + a_{\beta\gamma}^\alpha x^\beta x^\gamma - y^\alpha + a_{\beta\gamma}^\alpha y^\beta y^\gamma + a_{\beta\gamma}^\alpha x^\beta y^\gamma + 0(3).$$

For every pair of elements x and y we define their *commutator* u by

$$(7) \quad u = xyx^{-1}y^{-1}.$$

Hence

$$(7') \quad u^\alpha = \varphi^\alpha(\varphi(x, y), \varphi(x^{-1}, y^{-1})).$$

If x and y are infinitesimal, we can use (4) and (6) and obtain

$$\begin{aligned} u^\alpha &= x^\alpha + y^\alpha + a_{\beta\gamma}^\alpha x^\beta y^\gamma - x^\alpha + a_{\beta\gamma}^\alpha x^\beta x^\gamma - y^\alpha + a_{\beta\gamma}^\alpha y^\beta y^\gamma + a_{\beta\gamma}^\alpha x^\beta y^\gamma + \\ &\quad + a_{\beta\gamma}^\alpha (x^\beta + y^\beta)(-x^\gamma - y^\gamma) + 0(3) = a_{\beta\gamma}^\alpha x^\beta y^\gamma - a_{\beta\gamma}^\alpha y^\beta x^\gamma + 0(3) = \\ &= (a_{\beta\gamma}^\alpha - a_{\gamma\beta}^\alpha) x^\beta y^\gamma + 0(3). \end{aligned}$$

2. - The structure of the group.

We define the *structure constants* $C_{\beta\gamma}^\alpha$ of the group by

$$(8) \quad C_{\beta\gamma}^\alpha = a_{\beta\gamma}^\alpha - a_{\gamma\beta}^\alpha.$$

Then the above gives

$$(9) \quad u^\alpha = C_{\beta\gamma}^\alpha x^\beta y^\gamma + 0(3),$$

and of course we have

$$(10) \quad C_{\beta\gamma}^\alpha = -C_{\gamma\beta}^\alpha.$$

We see that for an Abelian group, for which $xy = yx$, the commutator vanishes. Therefore by (9) all the structure constants vanish for an Abelian group.

We next substitute the basic expansion (4) into the associative law (2) and set the coefficient of each product of the $x^\alpha, y^\alpha, z^\alpha$ independently equal to zero. All the coefficients of the first and second order products vanish automatically.

The condition that all the coefficients of the third order products vanish, gives a set of equations between the various $a_{\beta\gamma}^\alpha, g_{\beta\gamma\delta}^\alpha$ and $h_{\beta\gamma\delta}^\alpha$. It is possible to eliminate all the g 's and h 's and we then obtain the condition on the

$$C_{\beta\gamma}^\alpha = a_{\beta\gamma}^\beta - a_{\gamma\beta}^\alpha$$

as

$$(11) \quad C_{\beta\sigma}^\alpha C_{\gamma\delta}^\sigma + C_{\gamma\sigma}^\alpha C_{\delta\beta}^\sigma + C_{\delta\sigma}^\alpha C_{\beta\gamma}^\sigma = 0.$$

(Note: the three terms are obtained from the first by cycling the $\beta\gamma\delta$.)

LIE has shown that we do not need to worry about higher orders in the expansion of the associative law, because if the $C_{\beta\gamma}^\alpha$ satisfy (10) and (11), then there exists a set of φ -s satisfying (2) in all orders and such that u defined by (7') does have the development given in (9). The $C_{\beta\gamma}^\alpha$ are not sufficient to determine the φ completely. This corresponds to the arbitrariness in the choice of the parametrization. However, LIE has shown that groups having the same structure constants, are isomorphic in the neighbourhood of the unit element and at least homomorphic to a third group in the large. In other words, the $C_{\beta\gamma}^\alpha$ -s, i.e. the commutator of any two elements determine the structure of the group completely. This is *Lie's fundamental theorem*.

As examples of application, we consider the structures imposed on the group if some of the $C_{\beta\gamma}^\alpha$ vanish.

Let p be an integer $< r$, and agree to denote any index α which is $\leq p$ by a latin letter from the beginning of the alphabet, and any index α which is $> p$, by a latin letter from the end of the latin alphabet.

Example 1. - $C_{ab}^s = 0$ for all $a, b \leq p$ and $s > p$. Then the C_{ab}^d satisfy the conditions (10) and (11) for a group with p parameters. Clearly the elements of this group are the subset of the elements of the group with r parameters, having all parameters $x^\alpha = 0$ for $\alpha > p$. Thus we have a subgroup of the original group.

Example 2. - $C_{\alpha b}^s = 0$ for all $s > p, b \leq p$ and all α . Then from Example 1 we know that we have a subgroup whose elements y have $y^s = 0$. For any group element x the commutator with an y has $u^s = C_{\alpha\beta}^s x^\alpha y^\beta = C_{\alpha b}^s x^\alpha y^b = 0$.

Thus u belongs to the subgroup and so does $uy = xyx^{-1}$. That is, if y belongs to the subgroup, so does xyx^{-1} for all x , *i.e.* we have an invariant subgroup.

Example 3. — Suppose we have an invariant Abelian subgroup and the parametrization of the group is such that all elements x of the subgroup have $x^s = 0$ for $s > p$. Then from Example 2 we have that $C_{ab}^s = 0$. Further $C_{ab}' = 0$ because of the Abelian property.

Then consider the symmetric quadratic form

$$(12) \quad g_{\alpha\beta} = C_{\alpha\sigma}^{\rho} C_{\beta\rho}^{\sigma}.$$

If $\alpha = a \leq p$, then

$$g_{a\beta} = C_{a\sigma}^{\rho} C_{\beta\rho}^{\sigma}.$$

But $C_{a\sigma}^{\rho} = -C_{\sigma a}^{\rho} = 0$ if $\rho > p$; we can restrict the summation over ρ to one over $b \leq p$, thus

$$g_{a\beta} = C_{a\sigma}^b C_{\beta b}^{\sigma}.$$

Similarly, since $C_{\beta b}^{\sigma} = 0$ if $\sigma > p$, the σ summation can be replaced by one over $d \leq p$. Therefore

$$g_{a\beta} = C_{ad}^b C_{\beta b}^d.$$

But $C_{ad}^b = 0$, thus $g_{a\beta} = 0$, *i.e.* the first p rows of the matrix $g_{\alpha\beta}$ vanish; hence $\det g = 0$. Since any change of the parametrization, that is a co-ordinate transformation, multiplies $\det g$ by a finite non vanishing constant and therefore leaves the equation $\det g = 0$ invariant, it follows that if any invariant Abelian subgroup exists then $\det g = 0$. CARTAN proved that the converse is also true.

This result is of importance since we know that all reducible representations of a group are completely reducible unless there is an invariant Abelian subgroup. (A group whose all representations are completely reducible, *i.e.* one with $\det g \neq 0$ is called semisimple.)

3. — Representations of the group.

We now turn to the problem of finding the representations of the group, *i.e.* a set of matrices $D(x)$ with the property

$$D(x)D(y) = D(xy).$$

Consider elements $x(\tau)$ lying on a curve $x^{\alpha} = x^{\alpha}(\tau)$ passing through the origin, with $x^{\alpha}(0) = 0$. Define $\xi^{\alpha} \equiv (dx^{\alpha}/d\tau)_{\tau=0}$, so that $x^{\alpha}(\tau) = \tau\xi^{\alpha} + 0(\tau^2)$. Si-

milarly let $y-y(\tau)$ lie on another curve through the origin with $\eta^\alpha \equiv (dy^\alpha/d\tau)_{\tau=0}$ so that $y^\alpha(\tau) = \eta^\alpha \tau + 0(\tau^2)$. We may think on the ξ^α -s as components of a vector $\xi = (dx(\tau)/d\tau)_{\tau=0}$ which is along the tangent at the origin to the curve $x=x(\tau)$. We can define a set of basic vectors e_α , then $\xi = \xi^\alpha e_\alpha$, and similarly $\eta = \eta^\alpha e_\alpha$.

Consider

$$u^\alpha(\tau) = C_{\beta\gamma}^{\alpha} x^\beta y^\gamma + 0(3) = C_{\beta\gamma}^{\alpha} \xi^\beta \eta^\gamma \tau^2 + 0(3).$$

$u^\alpha(\tau)$ defines a third curve passing through the origin, having J as its tangent vector, where

$$J^\alpha = C_{\beta\gamma}^{\alpha} \xi^\beta \eta^\gamma.$$

Consider now $D(x(\tau))$. We have

$$\left(\frac{dD(x(\tau))}{d\tau} \right)_{\tau=0} = \left(\frac{\partial D}{\partial x^\alpha} \frac{dx^\alpha}{d\tau} \right)_{\tau=0} = \left[\frac{\partial D}{\partial x^\alpha} \cdot \frac{d}{d\tau} (\xi^\alpha \tau + 0(\tau^2)) \right]_{\tau=0} = \left(\frac{\partial D}{\partial x^\alpha} \right)_{x=I} \xi^\alpha \equiv D_\alpha \xi^\alpha,$$

where the

$$D_\alpha = \left(\frac{\partial D(x)}{\partial x^\alpha} \right)_{x=I}$$

are known as the *infinitesimal operators* of the representation.

Now expand $D(x)$ in powers of τ :

$$D(x) = 1 + \tau \xi^\beta D_\beta + \tau^2 E + 0(\tau^3).$$

Then determine $D(x^{-1})$ from the requirement $D(x)D(x^{-1}) = 1$ obtaining

$$D(x^{-1}) = 1 - \tau \xi^\beta D_\beta + \tau^2 (\xi^\beta D_\beta)^2 - \tau^2 E + 0(\tau^3).$$

Similarly

$$D(y) = 1 + \tau \eta^\gamma D_\gamma + \tau^2 E' + 0(\tau^3)$$

and

$$D(y^{-1}) = 1 - \tau \eta^\gamma D_\gamma + \tau^2 (\eta^\gamma D_\gamma)^2 - \tau^2 E' + 0(\tau^3).$$

Therefore

$$D(u) = D(x)D(y)D(x^{-1})D(y^{-1}) = 1 + \tau^2 [-\eta^\gamma D_\gamma \xi^\beta D_\beta + \xi^\beta D_\beta \eta^\gamma D_\gamma] + 0(\tau^3).$$

Thus

$$(13) \quad \left(\frac{dD(u)}{d\tau^2} \right)_{\tau=0} = \xi^\beta \eta^\gamma (D_\beta D_\gamma - D_\gamma D_\beta).$$

On the other hand, since

$$u^\alpha = C_{\beta\gamma}^\alpha \xi^\beta \eta^\gamma \tau^2 + 0(\tau^3)$$

we have:

$$\left(\frac{dD(u(\tau))}{d\tau^2} \right)_{\tau=0} = \left(\frac{\partial D}{\partial u^\alpha} \frac{du^\alpha}{d\tau^2} \right)_{\tau=0} = \left(\frac{\partial D}{\partial u^\alpha} \right)_{u=I} c_{\beta\gamma}^\alpha \xi^\beta \eta^\gamma = D_\alpha c_{\beta\gamma}^\alpha \xi^\beta \eta^\gamma.$$

By comparison with (13) we see that the D_α must satisfy the conditions

$$(14) \quad [D_\beta, D_\gamma] \equiv D_\beta D_\gamma - D_\gamma D_\beta = C_{\beta\gamma}^\alpha D_\alpha.$$

Thus we have shown that if we have a representation of a group, we can obtain a set of r constant matrices which satisfy a set of commutation rules defined by the structure constants. The antisymmetry of $[D_\beta, D_\gamma]$ in β and γ and the Jacobi identity

$$[D_\alpha [D_\beta, D_\gamma]] + [D_\beta [D_\gamma, D_\alpha]] + [D_\gamma [D_\alpha, D_\beta]] = 0,$$

when applied to (14), lead immediately to the conditions (10) and (11) on the $C_{\beta\gamma}^\alpha$, so that the restrictions on the structure constants are just those necessary to ensure a non-trivial solution of (14). Further, the inverse theorem also holds, i.e. if there is a set of r constant matrices satisfying (14), then they generate uniquely by integration a representation of the group. Thus the problem of finding an r -fold infinity of representative matrices reduces to that of finding r matrices. This problem was solved by CARTAN in 1913.

4. - Classification of representations.

We now attempt to obtain a set of classifying labels for the various irreducible representations of a semisimple group. For such a group $\det g \neq 0$, so that $g_{\alpha\beta}$ has an inverse, say $g^{\alpha\beta}$, with

$$g^{\alpha\beta} g_{\beta\gamma} = \delta_\gamma^\alpha.$$

We use the $g^{\alpha\beta}$ and $g_{\alpha\beta}$ for the raising and lowering of indices. Thus for instance

$$C_{\alpha\beta\gamma} \equiv g_{\alpha\delta} C_{\beta\gamma}^\delta = C_{\alpha\sigma}^\sigma C_{\delta\varrho}^\sigma C_{\beta\gamma}^\delta.$$

But by (11)

$$C_{\delta\varrho}^\sigma C_{\beta\gamma}^\delta = -C_{\delta\varrho}^\sigma C_{\gamma\varrho}^\delta = C_{\delta\gamma}^\sigma C_{\varrho\beta}^\delta,$$

therefore

$$C_{\alpha\beta\gamma} = C_{\alpha\sigma}^{\sigma} C_{\beta\delta}^{\sigma} C_{\gamma\varrho}^{\delta} - C_{\alpha\sigma}^{\sigma} C_{\gamma\delta}^{\sigma} C_{\beta\varrho}^{\delta}.$$

Thus

$$C_{\beta\alpha\gamma} = C_{\beta\sigma}^{\sigma} C_{\alpha\delta}^{\sigma} C_{\gamma\varrho}^{\delta} - C_{\beta\sigma}^{\sigma} C_{\gamma\delta}^{\sigma} C_{\alpha\varrho}^{\delta}.$$

In the first term of $C_{\beta\alpha\gamma}$ make the replacements $\sigma \rightarrow \varrho$, $\varrho \rightarrow \delta$, $\delta \rightarrow \sigma$, and in the second term make the replacements $\sigma \rightarrow \delta$, $\delta \rightarrow \varrho$, $\varrho \rightarrow \sigma$. Then one obtains

$$C_{\alpha\beta\gamma} = -C_{\beta\alpha\gamma}$$

and obviously this is true for any pair of indices, *i.e.* $C_{\alpha\beta\gamma}$ is totally anti-symmetric.

Then

$$C_{\gamma}^{\sigma\tau} = g^{\sigma\alpha} g^{\tau\beta} C_{\alpha\beta\gamma} = g^{\sigma\beta} g^{\tau\alpha} C_{\beta\alpha\gamma} = -g^{\tau\alpha} g^{\sigma\beta} C_{\alpha\beta\gamma} = -C_{\gamma}^{\tau\sigma},$$

therefore

$$(15) \quad C_{\gamma}^{\sigma\tau} + C_{\gamma}^{\tau\sigma} = 0.$$

We are now ready to look for the set of classifying labels.

For a given representation which has the infinitesimal operators D_{α} we form the operator F in this representation space, defined as

$$F = g^{\alpha\beta} D_{\alpha} D_{\beta},$$

F is known as the *Casimir operator*.

We can show that F commutes with all the D_{α} and consequently by integration with all the $D(x)$ of the representation. In fact:

$$\begin{aligned} [F, D_{\tau}] &= g^{\alpha\beta} [D_{\alpha} D_{\beta}, D_{\tau}] = g^{\alpha\beta} \{ D_{\alpha} [D_{\beta} D_{\tau}] + [D_{\alpha} D_{\tau}] D_{\beta} \} = \\ &= g^{\alpha\beta} \{ D_{\alpha} C_{\beta\tau}^{\gamma} D_{\gamma} + C_{\alpha\tau}^{\gamma} D_{\gamma} D_{\beta} \} = D_{\alpha} C_{\tau}^{\gamma\alpha} D_{\gamma} + C_{\tau}^{\gamma\beta} D_{\gamma} D_{\beta}. \end{aligned}$$

In the second term replace the dummy γ by α and β by γ ; we then obtain, using (15)

$$[F, D_{\tau}] = (C_{\tau}^{\gamma\alpha} + C_{\tau}^{\alpha\gamma}) D_{\alpha} D_{\gamma} = 0.$$

Thus F indeed commutes with all the matrices of the representation. Therefore, as we are considering an irreducible representation, by Schur's lemma

$$F = \lambda I.$$

Thus λ , the eigenvalue of the Casimir operator in the space of the irreducible representation under consideration, can be used as a label characterizing this representation.

The question arises whether or not all inequivalent irreducible representations have different λ . In general (contrast the rotation group in 3 dimensions) different representations may have the same λ , so that we must look for further labels.

One approach is to form further Casimir operators by means of a generalized $g_{\alpha\beta}$ defined as

$$\chi_{\alpha\beta\gamma\dots\kappa} = C_{\alpha\sigma}^{\epsilon} C_{\beta\tau}^{\sigma} \dots C_{\kappa\epsilon}^{\omega}.$$

We then raise all the indices on χ by means of the $g^{\mu\nu}$, and finally saturate all the indices on $\chi^{\alpha\beta\dots\kappa}$ with $D_{\alpha} D_{\beta} \dots D_{\kappa}$. Thus we form

$$F^{(K)} = \chi^{\alpha\beta\gamma\dots\kappa} D_{\alpha} D_{\beta} \dots D_{\kappa}.$$

It is easy to show that this also commutes with all the matrices of the representation under consideration, and so $F^{(K)}$ gives us a label $\lambda^{(K)}$.

Now the set $\{\lambda^{(K)}\}$ may or may not form an unequivocal characterization of all the inequivalent irreducible representations of the group. In the case of *simple* groups the set $\{\lambda^{(K)}\}$ is sufficient if for all the representations D and $(D^{-1})^x$ are equivalent representations. If this is not the case or the group is only semisimple, it is still possible to find further generalized Casimir operators whose eigenvalues are sufficient to give a complete characterization.

Four Dimensional Orthogonal Groups.

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1. - Basic notions and main theorem.

Consider a linear mapping of a four dimensional space x_i ($i = 1, 2, 3, 4$) onto itself

$$x_i = \sum_{k=1}^4 a_{ik} x_k,$$

where the coefficients are restricted by the orthogonality condition

$$\sum_{k=1}^4 a_{ik} a_{jk} = \delta_{ij},$$

which ensures that $x_1^2 + x_2^2 + x_3^2 + x_4^2$ remains invariant. It is easily seen that these transformations form a group, the 4-dimensional (complex) orthogonal group, denoted by O_4 . Making the restriction that all the a_{ij} are real, we obtain a subgroup R_4 , the four dimensional rotation group. Another subgroup is that of the homogeneous Lorentz transformations L , obtained by restricting a_{ik} to be real if both i and k are 1, 2, 3 or if $i = k = 4$, and otherwise to be pure imaginary. (Here x_1, x_2, x_3 are taken to be real and $x_4 = ict$ pure imaginary.)

A further subgroup is obtained by restricting a_{44} to be 1 and $a_{i4} = a_{4i} = 0$ ($i = 1, 2, 3$). This is the 3-dimensional (complex) orthogonal group O_3 . When further all the coefficients in O_3 are restricted to be real we obtain the subgroup R_3 of 3-dimensional rotations.

Our purpose is to study the structure of these various groups. We follow the treatment of EINSTEIN and MAYER (*Berl. Ber.* (1931)).

Consider a particular element of O_4 of the form

$$D = \begin{vmatrix} l & p & -n & m \\ -p & l & m & n \\ n & -m & l & p \\ -m & -n & -p & l \end{vmatrix},$$

where

$$l^2 + m^2 + n^2 + p^2 = 1.$$

D is the sum of a multiple of the unit matrix and a self-dual antisymmetric matrix. It is readily verified that D indeed belongs to O_4 . Further the product of any two D 's is again of the same type. (This assertion will be demonstrated in the following also indirectly).

Thus the D 's form a subgroup of O_4 .

Similarly we can consider another set of elements of O_4 of the form

$$A = \begin{vmatrix} \lambda & \pi & -\nu & -\mu \\ -\pi & \lambda & \mu & -\nu \\ \nu & -\mu & \lambda & -\pi \\ \mu & \nu & \pi & \lambda \end{vmatrix},$$

where

$$\lambda^2 + \mu^2 + \nu^2 + \pi^2 = 1.$$

This is the sum of a multiple of the unit matrix and an antiselfdual antisymmetric matrix. Everything we stated above for the D 's holds also for the A 's. Further, every D commutes with every A .

We assert that any member of O_4 may be written as the product of a D and a A and this factorization is unique up to a factor $+1$ or -1 in both D and A simultaneously. To prove this, we make use of the well known result that any finite element of O_4 can be obtained by successive application of infinitesimal transformations. (This holds naturally only if we restrict ourselves to transformations with $\det a = +1$).

The most general infinitesimal transformation of O_4 has the form

$$a_{ik} = \delta_{ik} + \varepsilon_{ik},$$

with $|\varepsilon_{ik}| \ll 1$ and $\varepsilon_{ik} = -\varepsilon_{ki}$.

Now take a D with $l=1$ and m, n and p infinitesimal and similarly a A with $\lambda=1$ and μ, ν and π infinitesimal. Then

$$(DA)_{ik} = (AD)_{ik} = \delta_{ik} + \varepsilon'_{ik},$$

where ε'_{ik} is clearly antisymmetric and

$$\varepsilon'_{ik} = D_{ik} + \Delta_{ik} \quad (\text{for } i \neq k).$$

It is easily seen that there always exists a set of infinitesimal m, n, p, μ, ν and π such that $\varepsilon'_{ik} = \varepsilon_{ik}$, namely

$$m = \frac{\varepsilon_{23} + \varepsilon_{14}}{2},$$

$$\mu = \frac{\varepsilon_{23} - \varepsilon_{14}}{2}, \quad \text{etc.}$$

Thus any infinitesimal element of O_4 is indeed the product of an infinitesimal D with an infinitesimal Δ , and since $[D, \Delta] = 0$, a sequence of infinitesimal transformations in O_4 generating an arbitrary finite element may be written

$$\prod_i (D^i \Delta^i) = \left(\prod_i D^i \right) \left(\prod_i \Delta^i \right) = D \Delta$$

proving our assertion.

Consequently it follows that the group O_4 is the direct product, of the group of the D 's with that of the Δ 's. Actually, due to the arbitrariness in the choice of ± 1 , this direct product runs over O_4 twice. Also, the group of the D 's is an invariant subgroup of O_4 . For, let $G = D\Delta$ be any element of O_4 . Then

$$GD'G^{-1} = D\Delta D' \Delta^{-1} D^{-1} = DD' \Delta \Delta^{-1} D^{-1} = DD' D^{-1} \equiv D'',$$

which belongs to the subgroup of D 's. Similarly, the group of the Δ 's is an invariant subgroup.

2. - Structure of O_4 and its subgroups.

We now go on to investigate the properties of the subgroups of the D 's and Δ 's. Knowing that O_4 is a non-trivial direct product of two groups, it follows that all the matrices of O_4 may be written after the application of a similarity transformation T as the direct (Kronecker) product of two 2×2 matrices; and that in particular after this transformation the D 's and Δ 's will have the form $1 \times S$ and $V \times 1$. This gives us enough information to find

the T . Such a T is for instance:

$$T = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 0 & i & 1 \\ 1 & -i & 0 & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & -i & 1 \end{vmatrix}.$$

Indeed

$$\begin{aligned} D' \equiv TDT^{-1} &= \begin{vmatrix} l+ip & 0 & m-in & 0 \\ 0 & l+ip & 0 & m-in \\ -(m+in) & 0 & l-ip & 0 \\ 0 & -(m+in) & 0 & l-ip \end{vmatrix} = \\ &= \begin{vmatrix} l+ip & m-in \\ -(m+in) & l-ip \end{vmatrix} \times \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = V \times 1, \end{aligned}$$

and

$$\begin{aligned} \Delta' \equiv T\Delta T^{-1} &= \begin{vmatrix} \lambda-i\pi & \mu+iv & 0 & 0 \\ -(\mu-iv) & \lambda+i\pi & 0 & 0 \\ 0 & 0 & \lambda-i\pi & \mu+iv \\ 0 & 0 & -(\mu-iv) & \lambda+i\pi \end{vmatrix} = \\ &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \times \begin{vmatrix} \lambda-i\pi & \mu+iv \\ -(\mu-iv) & \lambda+i\pi \end{vmatrix} = 1 \times S. \end{aligned}$$

It follows that any element $G = D\Delta$ of O_4 will go to the form $G' = D'\Delta' = (1 \times S) \cdot (V \times 1) = V \times S$ after the transformation T .

The conditions $l^2 + m^2 + n^2 + p^2 = 1$ and $\lambda^2 + \mu^2 + \nu^2 + \pi^2 = 1$ have the consequence that both S and V are unimodular matrices, *i.e.* $\det S = \det V = 1$.

(Incidentally, it is now trivial from the properties of the direct product to see that both the D 's and Δ 's form subgroups of O_4 and that $[D, \Delta] = 0$).

Summing up, as regards O_4 , we see that it is homomorphic to the direct product of the two-dimensional unimodular group C_2 with itself:

$$O_4 \sim C_2 \times C_2'.$$

This is a homomorphism and not an isomorphism because of the arbitrariness in the simultaneous signs of the matrices in C_2 and C_2' .

We now investigate the structure of the various subgroups of O_4 . To this

and we calculate explicitly the product of a D and a A :

$$DA = \begin{vmatrix} l\lambda + m\mu - & l\pi + p\lambda + & -(l\nu + n\lambda) + & m\lambda - l\mu + \\ -n\nu - p\pi & +m\nu + n\mu & +m\pi + p\mu & +n\pi - p\nu \\ \hline -(l\pi + p\lambda) + & l\lambda + n\nu - & l\mu + m\lambda + & n\lambda - l\nu + \\ +m\nu + n\mu & -m\mu - p\pi & +n\pi + p\nu & +p\mu - m\pi \\ \hline l\nu + n\lambda + & -(l\mu + m\lambda) + & l\lambda + p\pi - & p\lambda - l\pi + \\ +m\pi + p\mu & +n\pi + p\nu & -m\mu - n\nu & +m\nu - n\mu \\ \hline l\mu - m\lambda + & l\nu + p\mu - & l\pi - p\lambda + & l\lambda + m\mu + \\ +n\pi - p\nu & -m\pi - n\lambda & +m\nu - n\mu & +m\nu + p\pi \end{vmatrix}.$$

To obtain the subgroups R_4 , L , O_3 and R_3 we make restrictions on the l, m, \dots, ν, π .

For R_4 we must have all of these real. Then S and V both have the form

$$\begin{vmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{vmatrix} \quad \text{with} \quad |\alpha|^2 + |\beta|^2 = 1.$$

That is, the S and V matrices are now restricted to be also unitary, i.e. they both belong to the two-dimensional unitary unimodular group U_2 . Therefore

$$R_4 \sim U_2 \times U_2'.$$

Next if we take

$$\lambda = \bar{l}, \quad \mu = \bar{m}, \quad \nu = n, \quad \pi = \bar{p},$$

then DA has the form of an element of L . Since now $V = \bar{S}$, therefore C_2' is completely determined by C_2 so that

$$L \sim C_2.$$

Proceeding, we obtain O_3 by the restrictions

$$l = \lambda, \quad m = \mu, \quad n = \nu, \quad p = \pi.$$

Thus V and S are again not independent; therefore

$$O_3 \sim C_2.$$

We note in passing that this implies that O_3 and L are isomorphic.

Finally if we take all l, m, \dots, ν, π real and retain $l=\lambda \dots p=\pi$ we obtain R_3 . Then S and V are unitary unimodular matrices and related to each other; consequently

$$R_3 \sim U_2.$$

We emphasize the fact that R_4 and L have very different structures. In particular R_4 is homomorphic to the product of a group with itself, while L is homomorphic to a simple group.

3. - Representations.

This important difference is hidden when we pass to consideration of the representations of R_4 and L , for these turn out to have the same algebraic form, but for different reasons.

Considering first R_4 , it is known that one obtains all irreducible representations of a product group by taking Kronecker products of the irreducible representations of the component groups. The irreducible representations of U_2 are well known (of Wigner), these are the so called $\mathcal{D}^{(j)}$ representations with $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ of dimensionality $2j+1$. Thus the irreducible representations of R_4 have the form $\mathcal{D}^{(j)}(U_2) \times \mathcal{D}^{(j')}(U'_2)$.

Turning now to the irreducible representations of L , these are the irreducible representations of C_2 . These are obtained by considering linear substitutions onto the homogeneous polynomials of degree $2j$ in variables ξ and η , generated by

$$\begin{vmatrix} \xi' \\ \eta' \end{vmatrix} = C_2(\alpha\beta\gamma\delta) \begin{vmatrix} \xi \\ \eta \end{vmatrix} = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \begin{vmatrix} \xi \\ \eta \end{vmatrix},$$

where $\alpha\delta - \beta\gamma = 1$.

These representations are the $\mathcal{D}^{(j)}(C_2)$ and are analytic functions of $\alpha, \beta, \gamma, \delta$. But $\mathcal{D}^{(j)}(\bar{C}_2)$ also is a representation of C_2 and is not an analytic function of $\alpha, \beta, \gamma, \delta$ unless the matrix C_2 is unitary. It is known that in general the direct product of two representations $\mathcal{D}^{(j)}$ and $\mathcal{D}^{(j')}$ is reducible (Clebsch-Gordan theorem). But this is not true for $\mathcal{D}^{(j)}(C_2) \times \mathcal{D}^{(j')}(\bar{C}_2)$, since in the proof of this theorem use is made of differentiation with respect to the parameters of C_2 , but \bar{C}_2 is not an analytic function of C_2 . Thus

$$\mathcal{D}^{(j)} \left(\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \right) \times \mathcal{D}^{(j')} \left(\begin{vmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{vmatrix} \right) \equiv \mathcal{D}^{(j,j')}(\alpha, \beta, \gamma, \delta, \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}),$$

is indeed irreducible and is the most general irreducible representation of L . This is very similar in form to the irreducible representations of R_4 as given above. However this should not mask the essential difference in the nature of the groups L and R_4 .

Relativistic Invariance and Quantum Mechanics.

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In the following discussion the usual description of Q.M. in terms of state vectors in Hilbert space will be assumed. The object of the analysis is to obtain the maximum consequences of relativistic invariance under very general assumptions independent of the dynamical structure of the system.

This analysis is due to WIGNER [1].

There are two notions of state in Q.M.:

1) *Instantaneous state* describing the results of all observations at a particular instant.

2) *State sub specie aeternitatis* describing the results of all possible experiments during the whole history of the system (including perhaps experiments which cannot be performed instantaneously).

In the Heisenberg picture the state of the second type is described by a single vector normalized to one up to phase, a *unit ray*, in the Hilbert space. In the Schrödinger picture a family of vectors is needed.

Physically realizable unit rays are those which correspond to states which actually can be produced in the laboratory. This distinction is necessary because there are rays which are not physically realizable. For example, two vectors corresponding to states of different charge may not be added to give a ray corresponding to a physically realizable state (2).

(1) E. P. WIGNER: *Ann. Math.*, **40**, 149 (1939).

(2) C. G. WICK, E. P. WIGNER and A. S. WIGHTMAN: *Phys. Rev.*, **88**, 101 (1952).

In the following we shall use the second type of state exclusively.

To find the conditions for relativistic invariance we first discuss two notions of identity of physical systems:

1) *bodily identity*: the same systems is viewed by different observers A and B having different relations to the system,

2) *subjective identity*: A and B observe different systems in exactly the same way.

We have then the following situation:

Observer A	Observer B	
Co-ordinate system x_A^μ	Co-ordinate system x_B^μ	
Rays $\underline{\Phi}_A \in \mathcal{H}_A$	Rays $\underline{\Phi}_B \in \mathcal{H}_B$	\mathcal{H} = Hilbert space
Self-adjoint operators Q_A	Self-adjoint operators Q_B	

We must make a number of assumptions about what A and B can observe. We assume, for example, that the system does not affect the co-ordinate system (as it does *e.g.* in general relativity).

With respect to the concept of bodily identity we assume that every experiment of A is a possible one for B . Precisely, there exists a 1 — 1 mapping, $b_{B \leftarrow A}$, between physically realizable rays $\underline{\Phi}_A$ of an observer A onto those of any other observer B with the property:

$$(1) \quad (\underline{\Phi}_A, Q_A \underline{\Phi}_A) = (b_{B \leftarrow A} \underline{\Phi}_A, Q_B b_{B \leftarrow A} \underline{\Phi}_A),$$

where Q_B and Q_A represent bodily the same experiment.

The mapping b also satisfies:

$$(2) \quad b_{C \leftarrow A} = b_{C \leftarrow B} b_{B \leftarrow A}.$$

For the subjective identity, on the other hand, we assume that A and B find all possible states of the system to be the same *i.e.* all observers see the same world. Precisely, there exists a mapping $s_{B \leftarrow A}$ with the same properties as $b_{B \leftarrow A}$.

We obtain a relation between the mappings s and b by considering four observers A, A', B, B' . If A' is related to A by the same relativistic transformation g as B' to B , then states subjectively identical for A and B are also subjectively identical for A' and B' . Thus, we must assume the relation

$$s_{B' \leftarrow A'} = b_{B' \leftarrow B} s_{B \leftarrow A} b_{A \leftarrow A'},$$

or writing $B' = gB$ and $A' = gA$,

$$(3) \quad b_{B \leftarrow gB} s_{gB \leftarrow gA} = s_{B \leftarrow A} b_{A \leftarrow gA}.$$

The above discussion of relativistic invariance is based on the so-called « passive point of view » in contrast to the « active point of view » where a single observer is considered and to any state and any relativistic transformation there is associated a transformed state. In the latter case the relativistic invariance reduces to the requirement that relations are the same between the transformed states as between the original states. The active point of view is more convenient to work with and has the advantage of avoiding the concept of inverted observers; it will be adopted in the following.

We now prove the following

THEOREM. — $s_{A \leftarrow B} b_{B \leftarrow A}$ (which maps the state seen by A onto the state subjectively the same as that seen by B) defines a realization of the relativity group.

Proof. — We have to show that

$$(4) \quad (s_{A \leftarrow B} b_{B \leftarrow A})(s_{A \leftarrow C} b_{C \leftarrow A}) = s_{A \leftarrow D} b_{D \leftarrow A}$$

where $B = g_1 A$, $C = g_2 A$, and $D = (g_1 g_2) A$.

Now

$$b_{g_1 A \leftarrow A} s_{A \leftarrow g_2 A} = s_{g_1 A \leftarrow g_1 g_2 A} b_{g_1 g_2 A \leftarrow g_2 A},$$

from (3). Inserting this in the left hand side of (4) and using (2), we get the right hand side of (4).

The active interpretation of relativity transformations requires that for each $\underline{\Phi}_A = \underline{\Phi}$ there exists a corresponding $\underline{\Phi}_{g'}$, and $\underline{\Phi} \rightarrow \underline{\Phi}_{g'}$ is a transformation of physically realizable rays into themselves, which preserves physical relations between states. $\underline{\Phi}_{g'}$ is simply related to $\underline{\Phi}$ by the above mapping

$$\underline{\Phi}_{g^{-1}} = s_{A \leftarrow gA} b_{gA \leftarrow A} \underline{\Phi}.$$

In order to get rid of the observer and to have a single condition, we introduce transition probabilities $|(\underline{\Phi}, \underline{\Psi})|^2$, where $\underline{\Phi}$ and $\underline{\Psi}$ are Heisenberg-picture states. Then the condition of relativistic invariance (from the active point of view) can be simply formulated as

$$|(\underline{\Phi}, \underline{\Psi})|^2 = |(\underline{\Phi}_{g'}, \underline{\Psi}_{g'})|^2.$$

Let $\underline{\Phi} \rightarrow \underline{\Phi}_{g'} = \underline{T}_g \underline{\Phi}$. Then a relativistically invariant theory determines a

realization of the relativity group by ray correspondences on the physically realizable rays which preserves transition probabilities. We show that we can choose \underline{T}_g to be a linear unitary or antiunitary operator (this is not obvious), thus we deal with unitary (or antiunitary) representations up to factor.

We make the concept of physically realizable states mentioned above more precise by the so-called « assumption of commutative superselection rules » as follows: let the set of all observables be denoted $\{A\}$, then the set of all operators B (not necessarily observable) which commute with all A is denoted $\{A\}'$. $\{A\}'$ is assumed to be commutative. Then the Hilbert space breaks up into a direct sum of subspaces in which the superposition principle holds unrestrictedly (*coherent subspaces*). (Strictly speaking, we might get a direct integral rather than a sum. Here we specifically assume it is a direct sum).

Examples. — Charge superselection rule, baryon superselection rule. And one can derive new superselection rules from relativistic invariance, *e.g.* univalence superselection rule.

We make a final assumption: $|\underline{\Phi}, \underline{\Psi}_g\rangle|^2$ is continuous in g at the identity. The connection between ray transformations and the operators is then incorporated in the following fundamental theorem.

THEOREM. — « Given a Hilbert space \mathcal{H} and a ray correspondence $\underline{\Phi} \rightarrow \underline{T}\underline{\Phi}$ (defined for the unit rays) satisfying $|\underline{\Phi}, \underline{\Psi}|^2 = |\underline{T}\underline{\Phi}, \underline{T}\underline{\Psi}|^2$, then there exists a *unique* (up to a phase factor) operator T which is additive, *i.e.*, $T(\underline{\Phi}_1 + \underline{\Phi}_2) = T\underline{\Phi}_1 + T\underline{\Phi}_2$, and either unitary, *i.e.*, $(U\underline{\Phi}, U\underline{\Psi}) = (\underline{\Phi}, \underline{\Psi})$ or antiunitary, *i.e.*, $(U\underline{\Phi}, U\underline{\Psi}) = \overline{(\underline{\Phi}, \underline{\Psi})}$, which induces \underline{T} ».

(Proof see WIGNER's book, Appendix Ch. XX, and V. BARGMANN: *Ann. of Math.*, **59**, 1 (1954)).

Furthermore one can show that if $\underline{\Phi}$ lies in a fixed coherent subspace, then the corresponding $T\underline{\Phi}$ lies also in a single coherent subspace. (We note that two rays $\underline{\Phi}, \underline{\Psi}$ belong to the same coherent subspace if there exists a physically realizable ray $\underline{\chi}$ which is not orthogonal to either of them. Since \underline{T} preserves transition probabilities, $|\underline{\Phi}, \underline{\chi}|^2 \neq 0 \neq |\underline{\Psi}, \underline{\chi}|^2$ implies $|\underline{T}\underline{\Phi}, \underline{T}\underline{\chi}|^2 \neq 0 \neq |\underline{T}\underline{\Psi}, \underline{T}\underline{\chi}|^2$: so $\underline{T}\underline{\Phi}$ and $\underline{T}\underline{\Psi}$ lie in the same coherent subspace if $\underline{\Phi}$ and $\underline{\Psi}$ lie in the same coherent subspace.

Let us now take for simplicity the case where the coherent subspaces are labeled by 1, 2, Then T splits into

$$T(g) = \begin{pmatrix} T_{11}(g) & T_{12}(g) & \dots \\ T_{21}(g) & T_{22}(g) & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

There will be one non vanishing entry in each row, which is unitary or anti-

unitary. By continuity of $T(g)\Phi$ at the identity, *i.e.*, $T(0)=I$, the non vanishing terms are on the diagonal and unitary for the connected component of the relativity group. In any case, the multiplication law is

$$T(g_1)T(g_2) = \omega(g_1, g_2)T(g_1g_2),$$

where ω is of the form:

$$\omega(g_1, g_2) = \begin{vmatrix} \omega_1(g_1, g_2) & 0 & 0 & \dots & \dots \\ 0 & \omega_2(g_1, g_2) & 0 & \dots & \dots \\ 0 & 0 & \omega_3(g_1, g_2) & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

where $\omega_j(g)$ is a constant multiple of the identity.

Finally, we investigate the continuity of the operators:

THEOREM. — If T_g is continuous in the sense that $|\langle \Phi, T_g \Phi \rangle|^2$ is continuous, then $T(g)$ can be chosen as strongly continuous in a neighborhood of the identity. Every such choice arises from a continuous unitary representation up to a factor, of the connected component of the covering group of the relativity group (V. BARGMANN: *Ann. of Math.*, **59**, 1 (1954) or E. P. WIGNER: *Ann. of Math.*, **40**, 149 (1939)).

This leads us to study continuous unitary representations (up to a factor) of the connected components of the inhomogeneous unimodular group.

We consider from now on a single coherent subspace in which we have

$$(5) \quad U(g_1)U(g_2) = \omega(g_1, g_2)U(g_1g_2),$$

where $|\omega(g_1, g_2)|=1$. From a physical point of view a phase factor in the form $\exp[i\alpha(g)]U(g)$ has no effect. Such a phase factor will give rise in (5) to a new ω :

$$\omega' = \exp[i\{\alpha(g_1g_2) - \alpha(g_1) - \alpha(g_2)\}]\omega(g_1, g_2).$$

The following Theorem shows that we can reduce ω to ± 1 , or even get rid of it completely by passing to the covering group.

THEOREM. — « By permissible phase changes any unitary representation up to a factor of the connected component of the inhomogeneous Lorentz group, continuous in a neighborhood of the identity, can be converted into a representation up to a factor ± 1 ; and any such representation arises from a continuous unitary representation of the covering group ». (V. BARGMANN: *Ann. of Math.*, **59**, 1 (1954)).

To emphasize the significance of this theorem let us recall, in the following exercise, the corresponding situation for the Galilean group.

Exercise. — Prove that for a single Schrödinger particle of mass m whose relativity group is the Galilean group $\{\tau, \mathbf{a}, \mathbf{v}, R\}$ (τ = time translation, \mathbf{a} = space translation, \mathbf{v} = transformation to a moving co-ordinate, R = rotation) the factor $\omega(g_1, g_2) = \exp [i(m/\hbar)(\mathbf{v}_1 \cdot R_1 \mathbf{a}_2 + \frac{1}{2} v_1^2 \tau)]$ appears and cannot be removed by permissible phase changes. Note that this fact provides us with a superselection rule on the mass for the Galilean group: superposition of two states with different mass gives a state whose existence in Nature would contradict Galileian invariance (Bargmann's Superselection Rule).

1. — Continuous unitary representations of the inhomogeneous unimodular group.

1.1. *Representations of the translation subgroup; $U(a)$.* — As an example consider first the representation of $U(a)$ in the space L^2 .

$$(6) \quad (U(a)f)(x) = f(x - a),$$

where

$$\int |f(x)|^2 d^4x < \infty, \quad f \in L^2(R^4).$$

All irreducible unitary representations of $U(a)$ are one-dimensional (abelian group). Taking the Fourier transform of (6)

$$(U(a)\tilde{f})(k) = \exp[ik \cdot a] \tilde{f}(k).$$

Hence the representation reduces to a continuous sum labeled by k and we have: $U(a) = \int \exp[ik \cdot a] dk$ with the scalar product $\int |f(x)|^2 dx = \int |\tilde{f}(k)|^2 dk$ or:

$$(f, f) = \int (\tilde{f}, \tilde{f})_k d\mu(k).$$

Now in the general case, the Hilbert space is written as

$$\mathcal{H} = \int d\mu(k) \mathcal{H}(k),$$

where $\mathcal{H}(k)$ is the Hilbert space, belonging to the point k , of arbitrary dimension $\nu(k)$, finite, infinite or zero. Then $\Phi \in \mathcal{H}$ is equivalent to $\{\Phi(k)\}$, $\Phi(k) \in \mathcal{H}(k)$ for all k , and the scalar product is

$$(\Phi, \Psi) = \int d\mu(k) (\Phi(k), \Psi(k))_k.$$

In this space we can construct the most general unitary representation of the translation group.

THEOREM 1. — « Any continuous unitary representation of the translation group is unitary equivalent to a representation in the Hilbert space $\mathcal{H} = \int d\mu(k) \mathcal{H}(k)$ with the transformation law:

$$(7) \quad (U(a)\Phi)(k) = \exp[ik \cdot a] \Phi(k).$$

Two representations are equivalent if and only if 1) $\mu_1 = \mu_2$, 2) $v_1(k) = v_2(k)$ for almost all k . If A is a bounded operator in \mathcal{H} , commuting with a representation of the above form, then A has the form:

$$(A\Phi)(k) = A(k)\Phi(k).$$

Remark. — 1) $\mu_1 = \mu_2$ means $d\mu_1(k) = \varrho(k) \cdot d\mu_2(k)$, $\varrho(k) > 0$, i.e., essentially, the correspondence between k -values and states in Hilbert space is the same in both cases, since measure $d\mu(k)$ here indicates at what sets in momentum space there are no states in the Hilbert space. 2) $v_1(k) = v_2(k)$ means that there are the same number of states for each k in both cases.

1'2. Homogeneous Lorentz subgroup $U(0, A)$. — We shall split $U(0, A)$ into two parts. We define an operator $T(A)$ by

$$(8) \quad (T(A)\Phi)(k) = \Phi(A^{-1}k) \sqrt{\frac{d\mu(A^{-1}k)}{d\mu(k)}}.$$

Remark. — We shall drop the $\sqrt{}$ factor in (8) in the following. We see that this is permissible as follows.

From

$$U(0, A)U(a, 1)U(0, A)^{-1} = U(Aa, 1),$$

we find that $\mu(s) = \mu(As)$, where s is a set in k -space, and $v(k) = v(Ak)$ a.e. This allows us to identify the Hilbert spaces belonging to k and Ak . Then we use a theorem which says that a measure which has the property $\mu(s) = \mu(As)$ (a so-called *quasi invariant measure* under A) is actually equivalent to an invariant measure. We get the invariant measures by the formula:

$$\mu = c \delta(k) + \int_0^\infty d\varrho_+(m) d\Omega_m(k) + \int_0^\infty d\varrho_-(m) d\Omega_m(k) + \int_0^\infty d\varrho(im) d\Omega_{im}(k),$$

$$(k^2 = m^2, k_0 \geq 0) \quad (k^2 = m^2, k_0 \leq 0) \quad (k^2 = -m^2)$$

where $d\Omega_m(k)$ and $d\Omega_{im}(k)$ are the invariant volumes on the hyperboloids $k^2 = m^2$ and $k^2 = -m^2$ respectively.

Now $T(\mathcal{A})$ defined by (8) will satisfy by virtue of (7) and (8)

$$(9) \quad T(\mathcal{A})U(a, 1) = U(\mathcal{A}a, 1)T(\mathcal{A}).$$

We denote by $Q(\mathcal{A})$ the remainder of $U(0, \mathcal{A})$ after $T(\mathcal{A})$ has acted, *i.e.*,

$$(10) \quad Q(\mathcal{A}) = U(0, \mathcal{A})T(\mathcal{A})^{-1}.$$

Then the multiplication law for $U(0, \mathcal{A})$ gives

$$(11) \quad Q(\mathcal{A}_1)T(\mathcal{A}_1)Q(\mathcal{A}_2)T(\mathcal{A}_2) = Q(\mathcal{A}_1\mathcal{A}_2)T(\mathcal{A}_1\mathcal{A}_2).$$

From (9) and (10) we also find:

$$(12) \quad [Q(\mathcal{A}), U(a, 1)] = 0.$$

Hence by the second part of Theorem 1 we can write

$$(13) \quad (Q(\mathcal{A})\Phi)(k) = Q(k, \mathcal{A})\Phi(k).$$

Applying now (11) to a state Φ at the point k and using (13) we get the important formula:

$$(14) \quad Q(k, \mathcal{A}_1)Q(\mathcal{A}_1^{-1}k, \mathcal{A}_2) = Q(k, \mathcal{A}_1\mathcal{A}_2).$$

The problem is reduced now to the analysis of $Q(k, \mathcal{A})$. Eq. (14) is almost like a group product, but not quite. We note first that for \mathcal{A} 's which leave k invariant, *i.e.*, the little group L_k belonging to k , the correspondence $\mathcal{A} \rightarrow Q(k, \mathcal{A})$ is a representation of L_k . We shall show that this representation determines the representation of the full group up to unitary equivalence. Two representations are unitary equivalent if

$$(15) \quad U_2(a, \mathcal{A}) = VU_1(a, \mathcal{A})V^{-1}, \quad V \text{ unitary.}$$

The translation subgroup of U also satisfies

$$U_2(a, 1) = VU_1(a, 1)V^{-1}.$$

Then by Theorem 1 and the above Remark we may identify $U_1(a, 1)$ and $U_2(a, 1)$, $\mathcal{H}_1(k)$ and $\mathcal{H}_2(k)$, and write:

$$(V\Phi)(k) = V(k)\Phi(k).$$

Eq. (15) is now

$$Q_2(A)T(A) = VQ_1(A)T(A)V^{-1}$$

or

$$(16) \quad Q_2(k, A) = V(k)Q_1(k, A)V(A^{-1}k)^{-1}.$$

For the little group Eq. (16) indicates ordinary unitary equivalence. We can then discuss the consequences of unitary equivalence for Q_1 and Q_2 as follows. We start from representations of L_k , $Q_j(k, A)$, $j = 1, 2$, for a fixed k , and assume that they are unitary equivalent under $V(k)$, *i.e.*,

$$Q_2(k, A) = V(k)Q_1(k, A)V(k)^{-1}, \quad V(k): \mathcal{H}_k \rightarrow \mathcal{H}_k.$$

Now eq. (16) must be valid for all k and A . We continue the definition of $V(k)$ to other k -values by

$$(17) \quad V(A^{-1}k) = Q_2(k, A)^{-1}V(k)Q_1(k, A).$$

However, since there is, in general, more than one A which takes k into some point $A^{-1}k$ we must show that we get the same $V(k)$, *i.e.*,

$$V(A_1^{-1}k) = V(A^{-1}k) \quad \text{if} \quad A_1^{-1}k = A^{-1}k.$$

To show this we note that A and A_1 can differ only by a transformation in L_k . We write $A_1^{-1} = A^{-1}M$, $M \in L_k$, and

$$V(A_1^{-1}k) = Q_2(k, A_1)^{-1}V(k)Q_1(k, A_1).$$

From eq. (14) we can write

$$Q_2(k, A_1)^{-1} = Q_2(A^{-1}k, A^{-1})Q_2(k, M),$$

$$Q_1(k, A_1) = Q_1(k, M^{-1})Q_1(k, MA_1).$$

We insert these into (17) and get

$$V(A_1^{-1}k) = Q_2(k, A)^{-1} \frac{[Q_2(k, M)V(k)Q_1(k, M^{-1})]Q_1(k, A)}{V(k)} = V(A^{-1}k)$$

by virtue of (17).

We must further show that the definition (17) of $V(k)$ actually yields the relation

$$Q_2(q, A) = V(q)Q_1(q, A)V(A^{-1}q)^{-1}.$$

Let $q = \Sigma^{-1}k$, then

$$\begin{aligned} Q_2(q, A) &= [Q_2(k, \Sigma)^{-1} V(k) Q_1(k, \Sigma)] Q_1(\Sigma^{-1}k, A) [Q_1(k, \Sigma A)^{-1} V(k)^{-1} Q_2(k, \Sigma A)] \\ &= Q_2(k, \Sigma)^{-1} V(k) \underbrace{Q_1(k, \Sigma A) Q_1(k, \Sigma A)^{-1}}_I V(k)^{-1} Q_2(k, \Sigma A) \\ &= Q_2(k, \Sigma)^{-1} Q_2(k, \Sigma A) = Q_2(\Sigma^{-1}k, A). \end{aligned}$$

Thus we have solved the equivalence problem. Given a $V(k)$ at one point on the orbit, which mediates the unitary equivalence of the representations of the little groups there, we can extend its definition to all points of the orbit so that the relations implied by equivalence are satisfied everywhere on the orbit.

Now we pass to the actual construction of a set of $Q(k, A)$ satisfying the fundamental relation (14) for a given arbitrary representation of the little group L_k of fixed k . We use (14) to define $Q(k, A)$ for A not in L_k and arbitrary k . In (14), *i.e.*,

$$Q(k, A_1 A_2) = Q(k, A_1) Q(A_1^{-1}k, A_2)$$

we choose $A_1 \in L_k$ and A_2 in a set S such that if A_1 runs over L_k , $A_1 A_2$ runs over the whole group. If we form cosets of L_k then S consists of one element from each coset. In other words, A_2 runs over a parametrization of the cosets of L_k .

We choose $Q(k, A_2)$ arbitrarily for example as I , then (14) gives us a definition of Q for arbitrary $q = A^{-1}k$ and arbitrary A_2 :

$$(18) \quad Q(A^{-1}k, A_2) = Q(k, A)^{-1} Q(k, A A_2).$$

Since there are different Lorentz transformations A and M which take points k' and k'' into the same point we must again show the consistency of the above construction by proving that $Q(A^{-1}k, A_2) = Q(M^{-1}k, A_2)$. A and M can differ by an element of L_k . It suffices to consider the case $M \in S$. A can be written as $A = RM$, and MA_2 as $R'M'$, where R and $R' \in L_k$ and $M' \in S$. Then

$$Q(k, A) = Q(k, R) Q(k, M),$$

and

$$Q(k, A A_2) = Q(k, R R') Q(k, M').$$

Hence

$$\begin{aligned} Q(A^{-1}k, A_2) &= Q(k, M)^{-1} Q(k, R)^{-1} Q(k, R R') Q(k, M'), \\ &= Q(k, M)^{-1} Q(k, R') Q(k, M'), \\ &= Q(k, M)^{-1} Q(k, M A_2) = Q(M^{-1}k, A_2). \end{aligned}$$

The result of the whole analysis is the following theorem:

THEOREM. — « Every continuous unitary representation of the covering group of the connected component of the inhomogeneous Lorentz group is unitary equivalent to a representation of the form:

$$U(a, \Lambda)\Phi(k) = \exp[ika]Q(k, \Lambda)\Phi(\Lambda^{-1}k),$$

where $Q(k, \Lambda)$ is a family of unitary operators satisfying

$$Q(k, \Lambda_1)Q(\Lambda_1^{-1}k, \Lambda_2) = Q(k, \Lambda_1\Lambda_2)$$

and has the form given in (18), and where the Hilbert space in question is $\mathcal{H} = \int d\mu(k)\mathcal{H}(k)$. Two such representations are unitary equivalent if and only if: a) $\varrho_{\pm 1} = \varrho_{\pm 2}$, $\varrho_1(im) = \varrho_2(im)$; b) the representations of the little group are unitary equivalent for at least one (and therefore all) momentum vector on almost all orbits, for each family of continuous unitary representations of the little groups gives rise to a representation $\{a, \Lambda\} \rightarrow U(a, \Lambda)$.

Thus the representations of L_k and a measure on the momentum space determine the irreducible representations of the inhomogeneous Lorentz group. We can classify these latter representations as follows:

a) $\varrho^2 > 0$: we need a measure concentrated on the hyperbola $p^2 = m^2$. Take, for example, $p = (m, 0, 0, 0)$ (rest system), then L_k is isomorphic to R_3 . The irreducible representations will be labeled by $[m, j]$ of mass m and spin j with $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$

b) $p^2 = 0$: μ is concentrated on the light cone $p^2 = 0$. L_k reduces to the two-dimensional Euclidian group. Physically interesting irreducible representations are labeled by $0, j$, $j = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \dots$. Here j indicates the component of the spin along the momentum. \pm sign is the helicity (*).

c) $p = 0$: either trivial identity representation (vacuum), or infinite dimensional representation. See M. NAIMARK: *Linear Representations of the Lorentz Group*, in *Uspehi Mat. Nauk*, **9**, 19 (1954), *Am. Math. Soc. Trans.*, Vol. 6.

d) THEOREM. — « Let \mathcal{H} be a Hilbert space describing a coherent subspace of states. Let $\{a, \Lambda\} \rightarrow U(a, \Lambda)$ be a representation in \mathcal{H} of the covering group of the connected component of the inhomogeneous Lorentz group describing the transformation properties of the system. Then $\{a, \Lambda\} \rightarrow U(a, \Lambda)$ is of the following form:

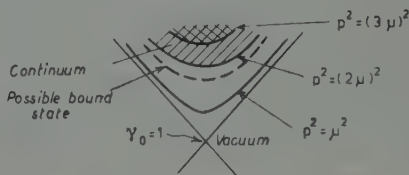
$$v_0 I + \int_0^\infty d\varrho_+(m) \sum_{j=0, \frac{1}{2}, 1, \dots} v([m, j])[m, j] + \sum_{j=0, \pm\frac{1}{2}, \pm 1, \dots} v([0, j])[0, j],$$

(*) For the remaining representations see E. P. WIGNER: *Zeits. f. Phys.*, **124**, 656 (1947).

where $\nu_0 = 0$ or 1, and $\nu([m, j])$, $\nu([0, j])$ are non-negative integer valued functions ».

Remark. — The above formalism says nothing about the interaction processes. As long as there are no bound states such processes do not change the equivalence classes $\{a, A\} \rightarrow U(a, A)$. Consider, for example, a Schrödinger particle with and without a potential (no bound states). The two Hamiltonians are unitary equivalent although very different physically. To describe the physics of interaction one has to adjoin to H , or to $U(a, A)$, other observables pertaining to the system.

Example. — The world of all possible numbers of neutral mesons of mass μ . The measure is concentrated as follows:



1'3. *Analysis of the inversions.* — We have to consider three types of inversion:

$$I \left(\begin{array}{ccc} I_s & , & I_t & , & I_{st} \\ \begin{array}{c} t \rightarrow t \\ x \rightarrow -x \end{array} & , & \begin{array}{c} t \rightarrow -t \\ x \rightarrow x \end{array} & , & \begin{array}{c} x \rightarrow -x \end{array} \end{array} \right).$$

They will be assumed to carry a coherent subspace into itself. We have to consider the factors ω in the relation

$$(19) \quad U(g_1)U(g_2) = \omega(g_1, g_2)U(g_1g_2)$$

which gives

$$(20) \quad U(I)^2 = \omega(I, I).$$

If U is unitary $\omega(I, I)$ can always be chosen equal to one by permissible phase changes. However, if U is antiunitary any phase change $U \rightarrow \exp[i\beta]U$ does not change the sign of U^2 , since in this case

$$(\exp[i\beta]U)(\exp[i\beta]U) = U^2.$$

Therefore, an anti-unitary transformation U makes a distinction between $U^2 = +1$ and $U^2 = -1$ and gives rise to a superselection rule, called *Type superselection rule*: In a coherent subspace $U^2 = +$ or $-$ throughout.

We shall show that $U(I_t)$ and $U(I_{st})$ are anti-unitary. Consequently, we get the following four *Types* (s is the spin):

Type	$(-1)^{2s}U(I_t)^2$	$(-1)^{2s}U(I_{st})^2$
I	+	+
II	+	-
III	-	+
IV	-	-

All known particles belong to Type I.

Let

$$(21) \quad U(I)U(g)U(I) = \chi(I, g)U(IgI),$$

where $\chi(I, g)$ is the proportionality constant, $g \in L_+$.

Then

$$U(I)U(g_1)\omega(I, I)U(g_2)U(I) = \chi(I, g_1)\chi(I, g_2)U(Ig_1I)U(Ig_2I)$$

or

$$(22) \quad \omega(I, I)\chi(I, g_1g_2) = \begin{cases} \frac{\omega(Ig_1I, Ig_2I)}{\omega(g_1, g_2)} \chi(I, g_1)\chi(I, g_2) & \text{if } U(I) \text{ is unitary} \\ \frac{\omega(Ig_1I, Ig_2I)}{\omega(g_1, g_2)} \chi(I, g_1)\chi(I, g_2) & \text{if } U(I) \text{ is anti-unitary.} \end{cases}$$

For the covering group the ratios in (22) reduce to one and we get

$$\chi(I, g_1g_2) = \omega(I, I)\chi(I, g_1)\chi(I, g_2),$$

i.e., $\omega(I, I)\chi(I, g)$ is a one-dimensional representation of the covering group. However there are no one-dimensional representations of the covering group except the identity. Hence

$$(23) \quad \chi(I, g) = \omega(I, I).$$

Let us look for the corresponding situation in the case of a Galilean group: the ratios in eq. (22) are 1 only if we take unitary space inversion and anti-unitary time inversion. Otherwise, it is impossible to satisfy the equation (22).

For a Lorentz group we cannot draw any such conclusion unless we make the physical assumption, that the energy is bounded below.

THEOREM. — « In a theory invariant under the connected component of the inhomogeneous L.G. in which the energy spectrum is bounded below, space

inversion must be represented by a unitary, and time inversion by an anti-unitary operator ».

Proof. — We start from

$$U(I)U(a, 1)U(I)^{-1} = U(Ia, 1)$$

and

$$U(a, 1) = \exp [iP_{\mu} a^{\mu}] .$$

Then

$$\begin{aligned} U(I)P^{\mu}U(I)^{-1} &= I^{\mu}_{\nu}P^{\nu} && \text{if } U(I) \text{ unitary} \\ -U(I)P^{\mu}U(I)^{-1} &= I^{\mu}_{\nu}P^{\nu} && \text{if } U(I) \text{ anti-unitary} \end{aligned}$$

or

$$\pm I^0_o(U(I)\Phi, P^0U(I)\Phi) = (\Phi, P^0\Phi)$$

which proves the Theorem.

Covariant Description of Polarization.

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1. - Description of one particle states.

We want to describe a particle of mass m , spin j . For this we use the irreducible representation $[m, j]$ described by WIGHTMAN. The wave function $\chi(p)$ has $2j+1$ components. The $Q(p, A)$ are $2j+1$ by $2j+1$ unitary matrices, such that

$$(U(a, A)\chi)(p) = \exp[ipa] Q(p, A)\chi(A^{-1}p),$$

$\chi(p)$ restricted to $\mathbf{p}^2 = m^2$ span the representation space of $[m, j]$ which is called the Hilbert space of the particle states. For a fixed \mathbf{p} , $\chi(p)$ span the Hilbert space of the polarization states for the particle of energy momentum \mathbf{p} .

2. - Mixtures of states.

If φ is any vector $\in \mathcal{H}$ one can construct the projection operator on the normalized state $|\varphi\rangle$ ($\langle\varphi|\varphi\rangle=1$)

$$P_\varphi = |\varphi\rangle\langle\varphi|; \quad P_\varphi|\psi\rangle = |\varphi\rangle\langle\varphi|\psi\rangle.$$

Characteristic properties are:

$$P_\varphi^2 = P_\varphi, \quad P_\varphi^* = P_\varphi, \quad \text{Tr } P_\varphi = \langle\varphi|\varphi\rangle = 1.$$

Expectation value of observable $A = \langle\varphi|A|\varphi\rangle = \text{Tr } A P_\varphi = \text{Tr } P_\varphi A$; note that $\varphi \rightarrow \exp[i\alpha]\varphi: P_\varphi \rightarrow P_\varphi$.

Now consider an incoherent mixture of orthogonal states q_n each with probability c_n ($0 \leq c_n \leq 1$; $\sum c_n = 1$).

The average value of an observable A over this mixture is:

$$\langle A \rangle = \sum c_n \langle q_n | A | q_n \rangle = \sum c_n \text{Tr } A P_{q_n} = \text{Tr } A \sum c_n P_{q_n} = \text{Tr } A \varrho = \text{Tr } \varrho A,$$

where we define

$$\varrho = \sum c_n P_{q_n}; \quad \text{Tr } \varrho = \sum c_n = 1 \quad \varrho^* = \varrho;$$

ϱ is called the density matrix of the mixture.

Note that for a true mixture $\text{Tr } \varrho^n < 1$ for each integer $n > 0$; $\varrho^n = \varrho$ for a pure state.

3. - Pure states and mixtures of polarization states.

We consider particles of non-vanishing mass. In the rest system: $p = (m, 0, 0, 0)$ there are $2j+1$ independent states. Therefore, under a rotation

$$\varrho \rightarrow \varrho' = D^{(j)} \varrho D^{(j)*},$$

where $D^{(j)}$ is the proper rotation matrix:

$$\varrho'_{\alpha\beta} = D^{(j)}_{\alpha\sigma} D^{(j)}_{\beta\tau} \varrho_{\sigma\tau},$$

which means that the elements of ϱ transform like the components of tensors under $D^{(j)} \otimes D^{(j)}$. Since $D^{(j)} \sim D^{(j)}$; $D^{(j)} \otimes D^{(j)} \sim D^{(2j)} + D^{(j-1)} + \dots + D^{(0)}$, which means that ϱ can be written as a sum of irreducible tensors of ranks 0, ... up to $2j$.

Since space inversion is known to commute with all rotations, by Schur's lemma, it is represented by a multiple of the unit matrix ($\exp[i\alpha]$); thus $D\bar{D}$ is invariant under space reflection, therefore the irreducible tensors are even under space reflection (*e.g.* one has a scalar $\text{Tr } \varrho$, a pseudovector...).

In order to describe a beam of particles one can normalize ϱ to $\text{Tr } \varrho = 1$ the intensity of the beam instead of unity: the beam is composed of particles which have all been prepared in the same way, so that one may consider that all particles in the beam are described by the same density matrix, the c_n 's being now the probabilities for finding particles in a given pure state. The use of a density matrix allows for the description of a system, the knowledge of which is incomplete.

Spin $\frac{1}{2}$:

$$D^{(\frac{1}{2})} \otimes D^{(\frac{1}{2})} = D^{(0)} + D^{(1)}.$$

The polarization state is described in terms of a scalar and a pseudo-vector \mathbf{S} :

$$\varrho = \frac{1}{2}(1 + \mathbf{S} \cdot \boldsymbol{\tau}), \quad \text{Tr } \varrho^2 = \frac{1 + \mathbf{S}^2}{2} \leq 1: \quad \mathbf{S}^2 \leq 1.$$

4. - Relativistic description of polarization.

As we know how to transform χ by any Lorentz transformation (see WIGHTMAN's lecture). (When Λ is in the little group Λ_p isomorphic to the rotation group, for mass \neq zero, one has exactly the ordinary published accounts of « non relativistic theory ».) The transformation law for the density matrix is:

$$\varrho'(p) = Q(p, \Lambda) \varrho(\Lambda^{-1} p) Q^*(p, \Lambda).$$

Example. Scattering of two particles: let $\varrho_1^{(i)}(p_1) \otimes \varrho_2^{(i)}(p_2)$ be the density matrix for uncorrelated initial particles: after the interaction the density matrix cannot in general be written as a tensor product of two density matrices, that is to say that, although the outgoing particles may not be individually polarized, there may be some correlation between their polarizations:

i.e. may be $\text{Prob} \begin{pmatrix} \uparrow \\ 1 \end{pmatrix} = \text{Prob} \begin{pmatrix} \downarrow \\ 1 \end{pmatrix}, \quad \text{Prob} \begin{pmatrix} \uparrow \\ 2 \end{pmatrix} = \text{Prob} \begin{pmatrix} \downarrow \\ 2 \end{pmatrix} \quad \text{but } \text{Prob} \begin{pmatrix} \uparrow & \uparrow \\ 1, 2 \end{pmatrix} \neq \text{Prob} \begin{pmatrix} \uparrow & \downarrow \\ 1, 2 \end{pmatrix}.$

5. - Particles of vanishing mass and finite spin.

The little group is isomorphic to the 2-dimensional Euclidean group. First of all one shall not consider the translations of this group since we want a finite spin; the remaining of the little group is then isomorphic with the group of rotations about \mathbf{p} and reflections through planes containing \mathbf{p} , that is to say the space group of diatomic molecules:

$$\mathcal{D}_j \otimes \mathcal{D}_j \sim \mathcal{D}_{2j} + \mathcal{D}_0^+ + \mathcal{D}_0^-$$

which correspond, to the well known classification $\Sigma^\pm, \pi, \Lambda, \Phi, \dots$

\mathcal{D}_0^- (scalar) corresponds to $\text{Tr } \varrho$ and describes the degree of polarization,

\mathcal{D}_0^+ (pseudoscalar) describes the circular polarization (helicity),

\mathcal{D}_{2j} (two dimensional vector *i.e.* an azimuth angle) describes the direction of the transverse polarization \mathbf{n} ($\mathbf{n} \cdot \mathbf{p} = 0, \mathbf{n}^2 = 1$). It can be written covariantly $\mathbf{p}^2 = 0 = \mathbf{p} \cdot \mathbf{n}, \mathbf{n}^2 = -1$; this \mathbf{n} is defined modulo a component along \mathbf{p} , *i.e.* $\mathbf{n} + \alpha \mathbf{p} \sim \mathbf{n}$.

6. - Representation by means of the Poincaré sphere.

Poles P_{\pm} represent pure circular polarization: $\xi = \pm 1$.

Equator E = transverse polarization. Any point of the sphere is of elliptical polarization $-1 \leq OG = \xi \leq 1$, $\mathbf{OM}^2 = -\mathbf{n}^2$. Pure state *i.e.* totally polarized $\xi^2 - \mathbf{n}^2 = 1$. Partially polarized state with degree of polarization η :

$0 \leq \eta = \sqrt{\xi^2 - \mathbf{n}^2} \leq 1$. Center of the sphere = unpolarized light.

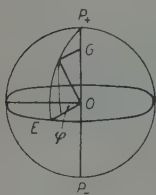


Fig. 1.

Exercise. By $\Lambda \in \Lambda_p$ (little group of p) the Poincaré sphere rotates around the axis of poles.

This is quite different of the case $m \neq 0$ spin $\frac{1}{2}$ where there is also only 2 linearly independent polarization states, but where by $\Lambda \in \Lambda_p$, S can be transformed in *any* S' of same length.

7. - Infinitesimal approach.

We shall now revert to the study of infinitesimal generators of the Lie algebra:

$$\begin{aligned} [P_\lambda, P_\mu] &= 0, \\ [M_{\mu\nu}, M_{\rho\sigma}] &= i[g_{\mu\rho}M_{\nu\sigma} - g_{\nu\rho}M_{\mu\sigma} - g_{\mu\sigma}M_{\rho\nu} + g_{\nu\sigma}M_{\rho\mu}], \\ [P_\lambda, M_{\mu\nu}] &= i[g_{\lambda\mu}P_\nu - g_{\lambda\nu}P_\mu]. \end{aligned}$$

The P 's and M 's are hermitian operators: as we saw they generate the envelopping associative algebra. The mathematician looks for a maximal abelian subalgebra.

The physicist has the same reflexes, but will call it a complete set of commuting observables.

We know already that $\mathbf{P}^2 = P_\mu P^\mu$ and $W^2 = W_\mu W^\mu$ belong to this set ($W_\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}$) since they belong to the center. One has

$$[P_\nu, W_\mu] = 0, \quad \text{but} \quad [W^\mu, W^\nu] = -i\varepsilon^{\mu\nu\rho\sigma}P_\rho W_\sigma.$$

Hence one can choose the four P_λ and W^2 and one W_μ . How to choose the last one « covariantly »? We shall now study a basis adapted to particle states of given energy momentum: *i.e.* ($P^\mu = p^\mu \in R^4$, $\mathbf{p}^2 = m^2$). To every such point p^μ of the spectrum of the P^μ 's there corresponds a Hilbert space of polarization states. Consider now the restriction of the preceeding operators to this Hilbert space:

$$\begin{aligned} P^\mu &\rightarrow p^\mu, \\ W^\mu &\rightarrow w^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}. \end{aligned}$$

Consider an orthogonal set of four-vectors: $\mathbf{u}^{(\alpha)} \cdot \mathbf{u}^{(\beta)} = g^{\alpha\beta}$:

(If $m \neq 0$ choose \mathbf{p}/m , \mathbf{n}_1 , \mathbf{n}_2 , \mathbf{n}_3 ; then \mathbf{n}_i are space like \mathbf{p}).

Then let $S^{(\alpha)} = W^\lambda n_\lambda^{(\alpha)}$; then $W_\lambda = g_{\alpha\beta} S^{(\alpha)} n_\lambda^{(\beta)} = S^{(\alpha)} n_{(\alpha)\lambda}$.

When $m \neq 0$, $W^\lambda = \sum_{i=1}^3 S^{(i)} n_{(i)}^\lambda$ because $W^\lambda p_\lambda = 0$

$$W^2 = W^\lambda W_\lambda = - \sum_{i=1}^3 S^{(i)2},$$

write now $\{S^i\} = \mathbf{S}$. The commutator of W_λ 's yields

$$\left[\frac{S^i}{m}, \frac{S^j}{m} \right] = i \epsilon^{ijk} \frac{S^k}{m},$$

which is to say that the little group is isomorphic to the three dimensional rotation group: hence

$$W^2 = -S^2 = -j(j+1)m^2.$$

Take furthermore any $u \in \mathbf{n} \cdot \mathbf{p} = 0$, then $\mathbf{W} \cdot \mathbf{n}$ has eigenvalues $-j < m < +j$. Thus we can choose for the center:

$$P_\lambda (P_\lambda P^\lambda = m^2), \quad W^2, \quad \mathbf{W} \cdot \mathbf{n}.$$

8. - Mass 0 case; $p^2 = 0$.

The only difference from the previous case is that one cannot choose an orthogonal basis which includes \mathbf{p} . We choose a time-like vector \mathbf{t} and complete it with \mathbf{n}_1 , \mathbf{n}_2 , $\alpha \mathbf{p} - \mathbf{t}$ to form an orthogonal basis $\mathbf{U}^{(i)}$. Writing $S^{(\alpha)} = W^\lambda U_\lambda^{(\alpha)}$, the condition $P_\lambda W^\lambda = 0$ yields

$$S^{(0)} + S^{(2)} = 0 \quad (\text{write } W_\lambda = S^{(\alpha)} U_\lambda^{(\alpha)}).$$

Furthermore

$$W^2 = S^{(0)2} - \sum_i S^{(i)2} = -S^{(1)2} - S^{(2)2}.$$

The commutation rules for the S 's are found to be

$$[S^{(3)}, S^{(1)}] = \frac{i}{\alpha} S^{(2)},$$

$$[S^{(2)}, S^{(3)}] = \frac{i}{\alpha} S^{(1)},$$

$$[S^{(2)}, S^{(1)}] = 0,$$

which characterizes the Lie algebra of L_p . This Lie algebra is isomorphic to that of the Euclidean two dimensional group: $S^{(1)}$, $S^{(2)}$ stand for the translation generators whereas $S^{(3)}$ stands for the rotation generator (rotations around \mathbf{p}).

Two cases must be distinguished:

$W^2 \neq 0$ which corresponds to the infinite spin case.

$W^2 = 0$: $(S^{(1)})^2 = (S^{(2)})^2 = 0$, i.e. since $(S^{(1)})^* = S^{(1)}$; $S^{(2)} = S^{(1)} = 0$.

The only generator left is $S^{(3)} = -S^{(0)}$, it defines an abelian group.

9. - Conclusion.

We have characterized the state of a particle of momentum \mathbf{p} , mass m ($j^2 = m^2$), spin j ($W^\mu W_\mu = -m^2 j(j+1)$) by means of a polarization operator $\mathbf{W} \cdot \mathbf{n}$, where \mathbf{n} is a unit vector orthogonal to \mathbf{p} . The eigenvalues of $\mathbf{W} \cdot \mathbf{n}$ are « magnetic » quantum numbers: $-j \leq m \leq +j$.

In the zero mass case, only two polarization states are available: $m = \pm j$ these states have opposite helicities.

Examples. Spin $\frac{1}{2}$.

Consider $\mathbf{n}^2 = -1$, $\mathbf{n} \cdot \mathbf{p} = 0$.

The projection operators on states for $\mathbf{W} \cdot \mathbf{n}/m = \pm \frac{1}{2}$ are $1 \pm ((2\mathbf{W} \cdot \mathbf{n}/m)/2)$. This is the density matrix of the pure state. More generally, we have seen for the density matrix

$$\frac{1 + \sum_i \varphi^i \tau^i}{2} = \frac{1 + 2 \sum_i \varphi^i \sigma^i}{2} \quad \text{where} \quad [\sigma^i, \sigma^j] = i\epsilon_{ijk} \sigma^k.$$

Here

$$\sigma^i = \frac{W^\lambda n_\lambda^{(i)}}{m} = \frac{\mathbf{W} \cdot \mathbf{n}^{(i)}}{m}$$

as we have seen.

Hence the most general density matrix, if we call $\sum \varphi^{(i)} \mathbf{n}^{(i)} \cdot \mathbf{s}$, is $(1 + (2\mathbf{W} \cdot \mathbf{s}/m))/2$ where the four-pseudovector \mathbf{s} is such that $\mathbf{s} \cdot \mathbf{p} = 0$, $0 \leq -\mathbf{s}^2 = \sum_i (\varphi^i)^2 = (\text{degree of polarization})^2 \leq 1$.

10. - Spin j .

The projection on the state polarized along \mathbf{n} with polarization $(-j \leq m \leq j)$ is

$$C_{(m)} \prod_{\lambda \neq m}^j (\mathbf{W} \cdot \mathbf{n} - \lambda) = P_{2j}(\mathbf{W} \cdot \mathbf{n}),$$

where $C_{(m)}^{-1} = (-1)^{j-m} (j-m)! (j+m)!$

It is a polynomial of degree $2j$ in $\mathbf{W} \cdot \mathbf{n}$.

More generally we have seen that the elements of the density *matrix* can be linearly combined into,

$$\frac{1}{2j+1} + \sum_i \varphi^i J^i + \sum_{ij} \varphi^{ij} J^i J^j + \sum_{ijk} \varphi^{ijk} J^i J^j J^k + \dots,$$

with $[J^i, J^j] = i\epsilon^{ijk} J^k$ and where the tensors $\varphi^i, \varphi^{ij}, \varphi^{ijk}$ transform under the little group (isomorphic to 3-dimensional rotation group) as D_1, D_2, \dots, D_{2j} . Hence the most general state of polarization will be covariantly described by the density matrix,

$$\frac{1}{2j+1} + s_\lambda W^\lambda + s_{\lambda\mu} W^\lambda W^\mu + s_{\lambda\mu\nu} W^\lambda W^\mu W^\nu + \dots + s_{\lambda\dots\sigma} W^\lambda \dots W^\sigma, \quad 2j \text{ indices.}$$

where $\sigma_{\lambda\dots\sigma}$ is a totally symmetric tensor such that $p^\lambda s_{\lambda\dots\sigma} = 0$, and $s_{\lambda\dots\mu\dots\sigma} = 0$ and other conditions on $s_{\lambda\dots\sigma} s^{\lambda\dots\sigma}$ fixing the polarization degree.

11. - Application to Dirac theory, mass $\neq 0$.

The Dirac amplitude $u(p)$ satisfying the Dirac equation $(p - m)u(p) = 0$ is transformed by inhomogeneous Lorentz transformations $(a, A \in L^\uparrow)$

$$(U(a, A)u(p) = \exp[ipa]S(A)U(A^{-1}p),$$

where S is a four dimensional representation of the homogeneous Lorentz group L^\uparrow . The $U(a, A)$ is a unitary operator in the Hilbert space of the $u(p)$ with the metric,

$$\int_{\mathcal{H}_m} |u^+(p)u(p)| d\Omega_m = \int |u^+(p)u(p)| \frac{d^3\mathbf{P}}{|p^0|} = \int |\bar{u}(p) \cdot A_i \gamma^i u(p)| \frac{d^3\mathbf{P}}{|p^0|^2}.$$

(The integration is over the hyperboloid $\mathbf{p}^2 = m^2 > 0$. The adjoint spinor $u^+ = \bar{u}A$ where the matrix A is defined by $A^* = A$, $\det A = 1$. $A\gamma^\mu A^{-1} = -\gamma^{\mu*}$; then $A^i \gamma^0$ is definite positive and is generally taken to be one.)

The infinitesimal operator $iM_{\mu\nu}$ is obtained by derivation of $(U(0, A)u)(p)$ at the identity, as shown in Section 4. We obtain

$$M_{\mu\nu} = \frac{1}{2} \sigma_{\mu\nu} + \frac{1}{i} \left(p_\mu \frac{\partial}{\partial p^\nu} - p_\nu \frac{\partial}{\partial p^\mu} \right),$$

where $\sigma^{\mu\nu} = (1/2i)[\gamma^\mu, \gamma^\nu]$. The two terms correspond to the spin part and orbital momentum part, the last part does not give contribution to W^λ .

We shall compute the restriction of W^λ on the (two dimensional) Hilbert space of polarization for energy momentum \mathbf{n}

$$W^\lambda = \frac{1}{2}\varepsilon^{\lambda\mu\nu\rho} p_\mu M_{\nu\rho} = \frac{1}{4}\varepsilon^{\lambda\mu\nu\rho} \sigma_{\nu\rho} p_\mu = \frac{1}{2}\gamma^5 \sigma^{\lambda\mu} p_\mu$$

(computation of $W^\lambda W_\lambda$ yields $\mathbf{W}^2 = -\frac{3}{4}m^2 = -j(j+1)m^2$ as expected). Now $W^\lambda n_\lambda = \mathbf{W} \cdot \mathbf{n} = \frac{1}{2}\gamma^5 \sigma^{\lambda\mu} n_\lambda p_\mu = \frac{1}{2}i\gamma^5 n\rho$.

We check that $(2\mathbf{W} \cdot \mathbf{n})/m = i\gamma^5 n(\rho/m)$ has square 1 and $\frac{1}{2}(1 + i\gamma^5 s(\rho/m))$ project the solution $u(p)$ of $(\rho - m)u(p) = 0$ onto the $u(p, s)$ Dirac amplitude for the state of energy momentum \mathbf{p} and total polarization along \mathbf{s} . Taking $Ai\gamma^0$, the projector

$$P(\mathbf{p}, \mathbf{s}) = \frac{1}{4} \left(1 + i\gamma^5 s \frac{\rho}{m} \right) (\rho + m) \frac{i\gamma^0}{|p^0|} = \frac{1}{4} (1 + i\gamma^5 s)(\rho + m) \frac{i\gamma^0}{|p^0|}.$$

is hermitian, and projects any Dirac spinor onto the spinor $u(p, s)$. Since it is a rank-one projector ($\text{Tr } P = 1$), one has therefore

$$P_{\alpha\beta}(p, s) = u_\alpha(p, s) \bar{u}_\beta(p, s) = u(p, s) \bar{u}(p, s) = P(p, s).$$

We have therefore explicitly constructed the density matrix of the pure polarization state p, s . For the general mixture $0 \leq -s^2 \leq 1$ is to be added to $\mathbf{s} \cdot \mathbf{p} = 0$.

We can choose for basis of states of polarization $u(p, \varepsilon s)$ with $\varepsilon = \pm 1$. Then any state vector is of the form $\sum_\varepsilon \chi_\varepsilon(p) u(p, \varepsilon s)$. (For a given p ; more generally, there can be an integration over p .)

The $\chi_\varepsilon(p)$ are those already studied at the beginning of V. We can therefore describe covariantly to our convenience the polarization states of spin $\frac{1}{2}$ either by a two by two density matrix

$$\varrho_{\varepsilon\eta}(p) = \chi_\varepsilon(p) \bar{\chi}_\eta(p)$$

or by the four by four density matrix $P_{\alpha\beta}(p, s)$.

The formula containing all the correspondence is (BOUCHIAT and MICHEL: *Nuclear Physics* (Feb. 1958)), using the usual Pauli matrix representation with τ_3 diagonal, ($(\tau_i)_{\varepsilon\eta}$ = elements of Pauli matrices)

$$u_\alpha(\mathbf{p}, \eta \mathbf{n}^{(3)}) \bar{u}_\beta(\mathbf{p}, \varepsilon \mathbf{n}^{(3)}) = \frac{1}{4} \left(1 + i\gamma^5 \left(\sum_i \eta^{(i)} (\tau_i)_{\varepsilon\eta} \right) \right) (\rho + m) \frac{i\gamma^0}{|p^0|}.$$

Hence, any field theoretical computation of polarization effects can be made covariantly and reduces to trace computation, although it is however possible to describe the computation in terms of the 2 by 2 , $\varrho_{e\eta}$ density matrices.

12. - Case of $m = 0$.

We do not explain here the limiting process given by MICHEL and WIGHTMAN, but indicate only the result:

$$P(p, s, \xi) = \frac{1}{4} (1 + i\gamma_5(s + \xi)) \rho \frac{i\gamma^0}{|p^0|},$$

where ξ , s have already been defined. We recall that

$$0 \leq \xi^2 - S^2 \leq 1, \quad \mathbf{P}^2 = \mathbf{p} \cdot \mathbf{s} = 0.$$

13. - Application to the Bargmann-Wigner theory of particles of arbitrary spin.

(BARGMANN-WIGNER: *Proc. Nat. Ac. of Sci.*, **34**, 211 (1948)).

We obtain, according to BARGMANN and WIGNER, the theory of particles of spin $j = n/2$ from the Dirac theory of spin $\frac{1}{2}$ by the following « transport of structure ».

Consider instead of the four dimensional Dirac spinor space \mathcal{E}_4 the n -th tensorial symmetric product $\overset{n}{V} \mathcal{E}_4$, that is the space of functions $u_{\alpha_1 \dots \alpha_n}(p)$, completely symmetrical in the α_i (*i.e.* invariant by any permutation of the indices $\alpha_1 \dots \alpha_n$), and satisfying the equations:

$$\sum_{\alpha_1 \dots \alpha_n} (p^{(k)} - m)_{\alpha'_1 \dots \alpha'_n, \alpha_1 \dots \alpha_n} u_{\alpha_1 \dots \alpha_n}(p) = (p^{(k)} - m)u(p) = 0,$$

where $p^{(k)} = i\gamma_\mu^{(k)} p^\mu$ and

$$(\gamma_\mu^{(k)})_{\alpha'_1 \dots \alpha'_n, \alpha_1 \dots \alpha_n} = \delta_{\alpha'_1 \alpha_1} \dots \delta_{\alpha'_{k-1} \alpha_{k-1}} (\gamma_\mu)_{\alpha'_k \alpha_k} \dots \delta_{\alpha'_n \alpha_n}$$

that is, in tensorial notation

$$\gamma_\mu^{(k)} = 1 \otimes 1 \otimes \dots \otimes \underset{k\text{-th place}}{\gamma_\mu} \otimes \dots \otimes 1 \quad (n \text{ terms in the product}),$$

The $u_{\alpha_1 \dots \alpha_n}(p)$ are transformed by the inhomogeneous Lorentz group according to

$$(U(\mathbf{a}, \Lambda)u)_{\alpha_1 \dots \alpha_n}(p) = \sum_{\beta_1 \dots \beta_n} \exp \left[\bigotimes_{i=1}^n i\mathbf{p} \cdot \mathbf{a} \right] [S(n)]_{\alpha_1 \dots \alpha_n, \beta_1 \dots \beta_n} u_{\beta_1 \dots \beta_n}(\Lambda^{-1}\mathbf{p}),$$

where $\bigotimes_{i=1}^n S(\Lambda) = S(\Lambda) \otimes \dots n \text{ times } \dots \otimes S(\Lambda)$.

(We leave to the reader the case of $a, A \in \mathcal{L}^\downarrow$ as we did for the Dirac case.)

This is a unitary representation of \mathcal{L}^\uparrow in the Hilbert space of scalar product (here $A = i\gamma^0$, and $Ai\gamma^0 = 1$)

$$\begin{aligned} (u'(p), u(p)) &= \int_{\mathcal{H}_m} |\bar{u}(p)(\otimes i\gamma^0) u(p)| \frac{d^3 \mathbf{p}}{|p^0|} = \int_{\mathcal{H}_m} \bar{u}(p) u(p) \frac{d^3 \mathbf{p}}{|p^0|^{n+1}} = \\ &= \sum_{\alpha_i} \int \bar{u}_{\alpha_1 \dots \alpha_n}(p) u_{\alpha_1 \dots \alpha_n}(p) \frac{d^3 \mathbf{p}}{|p^0|^{n+1}}. \end{aligned}$$

In a fashion analogous to the Dirac case, we find for the infinitesimal operator

$$M_{\mu\nu} = \sum_k \frac{1}{2} \sigma_{\mu\nu}^{(k)} + \frac{1}{i} \left(p_\mu \frac{\partial}{\partial p_\nu} - p_\nu \frac{\partial}{\partial p_\mu} \right),$$

where

$$\sigma_{\mu\nu}^{(k)} = \frac{1}{2i} [\gamma^{(k)}_\mu, \gamma^{(k)}_\nu].$$

Indeed the derivative of $\otimes S$ at the origin is

$$(S'(1) \otimes 1 \dots \otimes 1) + (1 \otimes (S'(1)) \otimes \dots \otimes 1) + \dots + (1 \otimes \dots \otimes S'(1)) = \sum_k S'_k.$$

Hence

$$W^\lambda = \frac{1}{4} e^{\lambda_{\mu\nu} p_\nu} \sum_k \sigma_{\nu\varrho}^{(k)} = \sum_k \frac{1}{2} \gamma^{(k)5} \sigma^{(k)\lambda\mu} p_\mu.$$

We leave to the reader to form the projector $P_{2j}(W \cdot n)$ already defined. Let us just conclude that the polarization states are covariantly described by the set of $2j$ tensors

$$s_\lambda, s_{\lambda\mu}, \dots, s_{\lambda\dots\sigma}, \dots$$

which are the expectation values of the products of 1 to $2j$, W_λ just computed. For pure states described by the amplitude $u(p)$

$$s_{\lambda\dots\sigma} = \bar{u}(p) W_\lambda \dots W_\sigma u(p) = \text{Tr } u(p) \otimes \bar{u}(p) W_\lambda \dots W_\sigma.$$

For the mixture described by the density matrix $P_{\alpha_1 \dots \alpha_n, \alpha_1 \dots \alpha_n}(p)$

$$s_{\lambda\dots\sigma} = \text{Tr } P(p) W_\lambda \dots W_\sigma.$$

Selected Problems in Renormalization Theory.

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(Lecture notes by S. GASIOROWICZ)

To bring out the similarity between quantum field theory and non-relativistic quantum mechanics it is instructive to approach the former in the historical manner, namely by imposing a generalization of the canonical quantization rules on the classical field operators. We start with a brief discussion of the

1. - Classical field theory and its quantization.

As in classical mechanics it is convenient to start with a Lagrangian function L which is written in the form of an integral over the Lagrangian density function

$$(1) \quad L = \int dx \mathcal{L}(x) \quad (dx = dx_1 dx_2 dx_3 dx_0)$$

For free scalar fields (*)

$$(2) \quad \mathcal{L}(x) = -\frac{1}{2} \left[\left(\frac{\partial \varphi}{\partial x_\mu} \right)^2 + m^2 \varphi^2 \right]$$

and the Euler-Lagrange equation of motion is

$$(3) \quad (\square - m^2)\varphi(x) = 0.$$

(*) We shall use natural units $\hbar = c = 1$ and the metric $x^2 = \bar{x}^2 - x_0^2$.

Given the value of the function $\varphi(x)$ and its time derivative $\dot{\varphi}(x)$ at all points at a given time,

$$(4) \quad \begin{cases} \varphi(\bar{x}, 0) = u(\bar{x}), \\ \dot{\varphi}(\bar{x}, 0) = v(\bar{x}), \end{cases}$$

the solution to the equation of motion is given by

$$(5) \quad \varphi(x) = - \int_{x'_0=0} d^3x' \left[\Delta(x-x'; m) v(\bar{x}') + \frac{\partial \Delta(x-x'; m)}{\partial x_0} u(\bar{x}') \right],$$

where

$$(6) \quad \Delta(x; m) = - \frac{i}{(2\pi)^3} \int dk \exp[ikx] \varepsilon(k) \delta(k^2 + m^2); \quad \varepsilon(k) = \frac{k_0}{|k_0|}.$$

The function $\Delta(x; m)$ is a solution of Eq. (3) and has the properties that

$$(7) \quad \Delta(x; m) = - \Delta(-x; m),$$

$$(8) \quad \Delta(x, m) = 0, \quad x^2 > 0, \quad (*)$$

$$(9) \quad \left[\frac{\partial}{\partial x_0} \Delta(x; m) \right]_{x_0=0} = - \delta(\bar{x}).$$

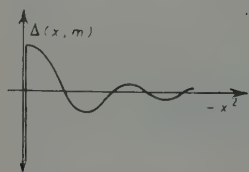


Fig. 1. - Plot of the function $\Delta(x, m)$ for $x_0 > 0$.

It can be computed explicitly, and is given by

$$(10) \quad \Delta(x; m) = - \frac{1}{2\pi} \varepsilon(x) \delta(x^2) + \frac{m^3 \varepsilon(x)}{4\pi} \frac{J_1(m\sqrt{-x^2})}{m\sqrt{-x^2}} \theta(-x^2).$$

To go over to the *quantum field theory* we use the Lagrangian to define the momentum $\pi(x)$ canonically conjugate to $\varphi(x)$

$$(11) \quad \pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(x)} = \dot{\varphi}(x)$$

and in analogy with the procedure in non-relativistic quantum mechanics we

(*) Such a condition is necessary to give invariant meaning to the symmetry condition (7).

impose the commutation rules

$$(12) \quad \begin{cases} [\pi(x), \varphi(x')] = -i \delta(\bar{x} - \bar{x}') & \text{if } x_0 = x'_0, \\ [\pi(x), \pi(x')] = [\varphi(x), \varphi(x')] = 0 & \text{if } x_0 \neq x'_0. \end{cases}$$

This procedure of replacing the classical Poisson bracket by i times the commutator leads to the Hamiltonian form of the equations of motion,

$$(13) \quad \begin{cases} i \dot{\pi}(x) = [\pi(x), H], \\ i \dot{\varphi}(x) = [\varphi(x), H]. \end{cases}$$

The Hamiltonian for the free field is

$$(14) \quad H = \int d^3x \mathcal{H}(x) = \int d^3x [\pi \dot{\varphi} - \mathcal{L}] = \frac{1}{2} \int d^3x [\pi^2 + (\nabla \varphi)^2 + m^2 \varphi^2].$$

To understand the meaning of the field operator $\varphi(x)$ it is useful to expand it in a Fourier series:

$$(15) \quad \varphi(x) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \left[\frac{1}{\sqrt{2\omega}} a(\vec{k}) \exp[i\vec{k}x] + \frac{1}{\sqrt{2\omega}} a^+(\vec{k}) \exp[-i\vec{k}x] \right].$$

Here $\omega = \sqrt{\vec{k}^2 + m^2}$ and V is the volume of the box in which the system is placed. At a later time we shall let this volume go to infinity, so that the sum over discrete momenta goes over into an integral according to the prescription

$$(16) \quad \frac{1}{V} \sum_{\vec{k}} \rightarrow \frac{1}{(2\pi)^3} \int d^3k.$$

A short calculation shows that the canonical commutation rules imply that

$$(17) \quad \begin{cases} [a(\vec{k}), a^+(\vec{k}')] = \delta_{\vec{k}\vec{k}'}, \\ [a(\vec{k}), a(\vec{k}')] = [a^+(\vec{k}), a^+(\vec{k}')] = 0. \end{cases}$$

Furthermore one finds that

$$(18) \quad H = \sum_{\vec{k}} \omega(a^+(\vec{k}) a(\vec{k}) + \frac{1}{2}) = \sum_{\vec{k}} H_{\vec{k}}.$$

The free field thus corresponds to a sum of harmonic oscillators. We know

that there is a ground state $|0\rangle$ with the property that

$$(19) \quad a(k)|0\rangle = 0.$$

Further

$$(20) \quad a^+(k)|0\rangle = |k\rangle$$

such that

$$(21) \quad H_k|k\rangle = \frac{3}{2}\omega|k\rangle.$$

It at first appears troublesome that

$$(22) \quad H_k|0\rangle = \frac{1}{2}\omega|0\rangle$$

as this would imply an infinite energy for the vacuum. It is clear however that only energy differences are observable, and the problem of infinite vacuum energy is disposed of by redefining the Hamiltonian to be

$$(23) \quad H' = \sum_k H'_k = \sum_k \omega a^+(k) a(k).$$

This is the first example of a renormalization, less trivial examples of which will appear later.

For free fields the commutators at arbitrary times may be evaluated by direct calculation (*). They are summarized by

$$(24) \quad [\varphi(x), \varphi(y)] = i \Delta(x-y; m).$$

It is only in the case of free fields that this commutator is a *c*-number. Such is not the case for the anticommutator $\{\varphi(x), \varphi(y)\}$. However the vacuum expectation value of this quantity is of course a *c*-number and since it will be of interest in specific calculations we write it down:

$$(25) \quad \langle 0 | \{\varphi(x), \varphi(x')\} | 0 \rangle = \Delta^{(1)}(x'-x; m) - \frac{1}{(2\pi)^3} \int d^3k \exp[ik(x'-x)] \delta(k^2 + m^2).$$

(*) Or by using

$$\varphi(x) = - \int_{y_0=x_0} d^3y \left[\Delta(x-y; m) \frac{\partial}{\partial y_0} \varphi(y) + \frac{\partial}{\partial x_0} \Delta(x-y; m) \varphi(y) \right]$$

and the equal time commutators.

An explicit form for $\Delta^{(1)}(x; m)$ is

$$(26) \quad \Delta^{(1)}(x; m) = -\frac{m^2}{4\pi} \frac{N_1 m(\sqrt{-x^2})}{m\sqrt{-x^2}}.$$

Note that $\Delta^{(1)}(x; m)$ does not vanish outside the light-cone, and is more singular than $\Delta(x; m)$ for $x^2 = 0$.

For future use we write down the corresponding results for the free electromagnetic field and for the free spinor field.

Electromagnetic field:

$$(27) \quad \square A_\mu(x) = 0,$$

$$(28) \quad [A_\mu(x), A_\nu(x')] = i\delta_{\mu\nu}\Delta(x-x'; 0) = i\delta_{\mu\nu}D(x-x'),$$

$$(29) \quad \langle 0 | \{A_\mu(x), A_\nu(x')\} | 0 \rangle = \delta_{\mu\nu}\Delta^{(1)}(x-x'; 0) = \delta_{\mu\nu}D^{(1)}(x-x') \quad (*) .$$

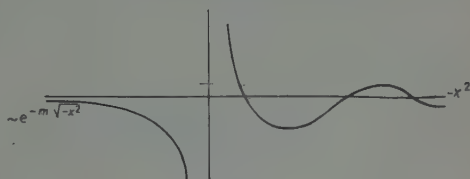


Fig. 2. - Plot of the function $\Delta^{(1)}(x; m)$

Spinor field:

$$(30) \quad \left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m \right)_{\alpha\beta} \psi_\beta(x) = 0, \quad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu},$$

$$(31) \quad \{\bar{\psi}_\alpha(x), \psi_\beta(x')\} = -iS_{\beta\alpha}(x'-x; m),$$

$$(32) \quad \langle 0 | [\bar{\psi}(x), \psi(x')] | 0 \rangle = S^{(1)}(x'-x; m),$$

where

$$(33) \quad S(x; m) = \left(\gamma \frac{\partial}{\partial x} - m \right) \Delta(x; m), \quad \text{etc. and} \quad \bar{\psi} = \psi^* \gamma_4.$$

It is still necessary to give a physical meaning to the quantized fields. These differ from the classical fields only in their non-commutativity, which however has a physical interpretation. As in non-relativistic quantum mechanics an operator relation of the type

$$(34) \quad [A, B] = c \quad (\text{a } c\text{-number})$$

implies that the quantities A, B cannot be simultaneously specified with an

(*) In the case of this field, because of the freedom corresponding to gauge transformations the derivation of these relations is not completely straightforward. However if one works with the Gupta-Bleuler gauge (indefinite metric) these results are obtained.

accuracy greater than expressed by

$$(35) \quad \Delta A \cdot \Delta B \gtrsim c,$$

where ΔA and ΔB are the uncertainties in the measured values of A and B . The application of this result to quantum field theory meets with difficulties. An examination of eq. (24) shows for instance that although in many cases the usual interpretation is in accordance with what one would expect on physical grounds, the situation when the fields are taken at the same space-time point or at points separated by light-like distances is so singular as to require a revision of this interpretation of the commutation relations. This difficulty was discussed by BOHR and ROSENFELD in their paper on the measurability of the electromagnetic field. They pointed out that the field at a point is an idealization not susceptible of measurement because of the finite size of the classical measuring apparatus and the finite times necessary to determine forces through their effect on the velocities of macroscopic test bodies. Thus only averages of field quantities over finite space-time regions are to be considered as measurable, and only the commutation relations in the form

$$(36) \quad [\varphi(R), \varphi(R')] = \frac{1}{RR'} \int_R \int_{R'} dx' i \Delta(x - x'; m),$$

where

$$(37) \quad \varphi(R) = \frac{1}{R} \int_R dx \varphi(x),$$

should be interpreted in accordance with the uncertainty principle. The analysis of eq. (36) by Bohr and Rosenfeld (carried out for the electromagnetic field) established that only when the regions R and R' were time-like (or light-like) with respect to each other could the measurements interfere; in the case of space like separations, where no signal can connect the two regions, the fields could be measured simultaneously. The result of this examination was to show that the formalism of quantum electrodynamics was in complete accordance with the physical principles of the special theory of relativity and the quantum theory for the test bodies. A similar analysis for fields other than the electromagnetic field can, of course, formally be carried through, but it makes very little sense as there are no macroscopic « test bodies » with « mesonic charge » available in nature. Nevertheless the same mathematical formalism which is used in quantum electrodynamics is copied with only minor modifications in meson theory. The implicit assumption that quantities like $(1/R) \int_R dx \varphi(x)$ are measurable is a perhaps completely unjustified extrapolation from quantum electrodynamics.

2. - A scalar field in interaction with a c -number source.

As a preliminary to the treatment of two interacting fields, it is very useful to consider the problem of a field generated by a classical distribution. Still keeping to the scalar field for simplicity, we have the equation of motion

$$(38) \quad (\square - m^2)\varphi(x) = -j(x).$$

The solution is

$$(39) \quad \varphi(x) = \varphi^{\text{in}}(x) + \int dx' \Delta_R(x - x'; m) j(x').$$

$\Delta_R(x; m)$ is the « Green's function » for the problem, which satisfies the equation

$$(40) \quad (\square - m^2) \Delta_R(x; m) = -\delta(x).$$

A representation of $\Delta_R(x; m)$ is

$$(41) \quad \Delta_R(x, m) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{(2\pi)^4} \int dp \frac{\exp[ipx]}{p^2 - (p_0 + i\varepsilon)^2 + m^2},$$

where the choice of path of integration is such as to make $\Delta_R(x; m)$ vanish for $x_0 < 0$. Actually

$$(42) \quad \begin{cases} \Delta_R(x, m) = -\Delta(x, m) & x_0 > 0 \\ = 0 & x_0 < 0 \end{cases}$$

and this feature means that the integration in eq. (39) is over the past light-cone.

The operator $\varphi^{\text{in}}(x)$ is a free field operator obeying the homogeneous equation and the free field commutation relations. As $x_0 \rightarrow -\infty$ the light-cone, crudely speaking, recedes into the distant past, and thus

$$(43) \quad \varphi(x) \rightarrow \varphi^{\text{in}}(x).$$

(This is the reason for the « in » superscript). The integral on the r.h.s. of eq. (39) is only well-defined if $j(x') \rightarrow 0$ as $x'_0 \rightarrow -\infty$. To show that the solution (39) is all that is necessary, we must prove that $\varphi(x)$ diagonalizes the Hamiltonian H for this problem,

$$(44) \quad H(\varphi) = H_c(\varphi) + H_1(\varphi) = \frac{1}{2} \int d^3x [\pi^2(x) + (\nabla\varphi(x))^2 + m^2\varphi^2(x)] - \int d^3x \varphi(x) j(x).$$

If we substitute the solution (39) into the Hamiltonian, we find after a short calculation that

$$(45) \quad H = H_0(\varphi^{\text{in}}) + E + \\ + \int d^3x \int d^3x' \frac{\partial j(x')}{\partial x'_0} \left\{ \Delta_R(x-x', m) \frac{\partial \varphi^{\text{in}}(x')}{\partial x'_0} - \frac{\partial}{\partial x_0} \Delta_R(x-x'; m) \varphi^{\text{in}}(x') \right\},$$

where E is a c -number. The first two terms have eigenstates generated by the application of free field creation operators (in-operators) to the vacuum. These states do not however form eigenstates of the total Hamiltonian because the term linear in $\varphi^{\text{in}}(x)$ causes transitions between states having different occupation numbers of incoming particles. This is not unexpected because with a time-dependent source the energy is not a constant of the motion. If the source is time independent the term linear in $\varphi^{\text{in}}(x)$ is seen to vanish and, the $|\text{in}\rangle$ states are eigenstates of the total Hamiltonian. With a time independent source, the integral in the solution (39) is not well defined, and a limiting procedure is necessary to give it meaning. A way of doing this is to define a solution

$$\varphi(x, \alpha) = \varphi^{\text{in}}(x) + \int d^3x' \Delta_R(x-x'; m) j(x') \exp[-\alpha |x'_0|],$$

with α a small positive parameter, which is allowed to go to zero at the end of the calculation. This method of «adiabatic switching off of the interaction» has been shown to give a well defined limit in problems with a finite number of degrees of freedom as well as in perturbation calculations to all orders. In the problem under consideration it gives a well defined limit. It should be noted that because of the time dependence of the source the energy changes adiabatically from $x_0 = -\infty$ to the present.

In our problem

$$(46) \quad \begin{cases} H_0(\varphi^{\text{in}}) |\text{in}\rangle = E_n^{(0)} |\text{in}\rangle \\ H(\varphi) |\text{in}\rangle = (E_n^{(0)} + E) |\text{in}\rangle \end{cases}$$

so that the $|\text{in}\rangle$ states remain eigenstates of the Hamiltonian but the energy changes by what is essentially the self energy of the source.

It is important to notice the difference between the $|\text{in}\rangle$ states which obey

$$(47) \quad H_0(\varphi^{\text{in}}) |\text{in}\rangle = E_n^{(0)} |\text{in}\rangle$$

and the «bare particle» states (or Tamm-Dancoff states) which are defined by

$$(48) \quad H_0(\varphi) |TD\rangle = E_n |TD\rangle.$$

In our problem we may explicitly expand the $|in\rangle$ states in terms of TD states. We find that for example

$$(49) \quad |0, in\rangle = \sum C_n |n; TD\rangle = c \prod_{\vec{q}} \sum_{n=0}^{\infty} \left(\frac{-j(\vec{q})}{\sqrt{2}\omega^3} \right)^n \frac{1}{\sqrt{n!}} |n \times \vec{q}; TD\rangle,$$

where $|n \times q; TD\rangle$ represents an n -fold occupied TD state and $j(\vec{q})$ is the Fourier transform of the current. The normalization coefficient is

$$(50) \quad |c|^2 = \exp \left[-\frac{1}{2} \sum_{\vec{q}} \left| \frac{j(\vec{q})}{\omega} \right|^2 \right].$$

As the source size approaches zero, *i.e.* $j(\vec{q}) \rightarrow \text{const}$:

$$(51) \quad |c|^2 \rightarrow 0,$$

which means that the probability of finding a bare particle state in a given physical state approaches zero. The Tamm-Dancoff approximation in which physical states are approximated by only a few terms in an expansion of the type (49) may thus be expected to give unreliable results whenever the source size is small.

It may be noted that in the point source limit the expansion (49) becomes more or less meaningless. This phenomenon was first pointed out by VAN HOVE. We want to point out that the non-existence of this expansion does not seem to have any bearing on the physical consistency of the model. All reasonable questions can be answered in terms of the physical states, without any reference to the TD states.

As an illustration of some of the things mentioned earlier, also because of the appearance of some interesting new features it is instructive to work out a special example in perturbation theory, and to stay as close as possible to physics we shall choose an example from quantum electrodynamics.

3. - Calculation of vacuum polarization by an external field.

The equations of quantum electrodynamics are

$$(52) \quad \begin{cases} \left(\gamma_{\mu} \frac{\partial}{\partial x_{\mu}} + m \right) \psi(x) = ie \gamma_{\mu} (A_{\mu}(x) + A_{\mu}^{\text{ext}}(x)) \psi(x) \equiv f(x), \\ \square A_{\mu}(x) = -\frac{ie}{2} [\bar{\psi}(x), \gamma_{\mu} \psi(x)] = -j_{\mu}(x) \end{cases}$$

and their formal solutions are

$$(53) \quad \begin{cases} \psi(x) = \psi^{\text{in}}(x) - \int S_R(x-x') f(x') \, dx' , \\ A_\mu(x) = A_\mu^{\text{in}}(x) + \int D_R(x-x') j_\mu(x') \, dx' , \end{cases}$$

where

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m \right) \psi^{\text{in}}(x) = 0 \{ \bar{\psi}^{\text{in}}(x'), \psi^{\text{in}}(x) \} = -i S(x'-x) \quad \text{etc.}$$

The only way we have of treating such coupled equations is by an expansion in powers of e (and the external field $A_\mu^{\text{ext}}(x)$). Thus

$$(54) \quad \psi(x) = \psi^{\text{in}}(x) - \int S_R(x-x') [ie\gamma_\mu A_\mu^{\text{in}}(x') + ie\gamma_\mu A_\mu^{\text{ext}}(x')] \psi^{\text{in}}(x') \, dx' + \dots$$

The current operator $j_\mu(x)$ is given by

$$(55) \quad j_\mu(x) = \frac{ie}{2} [\bar{\psi}^{\text{in}}(x), \gamma_\mu \psi^{\text{in}}(x)] + \frac{1}{2} e^2 \int dx' \{ [\bar{\psi}^{\text{in}}(x), \gamma_\mu S_R(x-x') \gamma_\nu \psi^{\text{in}}(x')] + [\bar{\psi}^{\text{in}}(x) \gamma_\nu S_A(x'-x), \gamma_\mu \psi^{\text{in}}(x)] \} (A_\nu^{\text{in}}(x') + A_\nu^{\text{ext}}(x')) .$$

In particular

$$(56) \quad \langle 0 | j_\mu(x) | 0 \rangle = \frac{e^2}{2} \int dx' \{ \text{Sp} [\gamma_\mu S_R(x-x') \gamma_\nu S^{(1)}(x'-x)] + \text{Sp} [\gamma_\nu S_A(x'-x) \gamma_\mu S^{(1)}(x-x')] \} A_\nu^{\text{ext}}(x') .$$

Due to the presence of the external field the vacuum expectation of the current does not vanish. Physically, because of the possibility of pair creation the vacuum behaves like a polarizable medium, and the application of an external field gives rise to a polarization current which we wish to calculate.

We rewrite (56) in the form

$$(57) \quad \langle 0 | j_\mu(x) | 0 \rangle = \int dx' K_{\mu\nu}(x-x') A_\nu^{\text{ext}}(x') .$$

In all such calculations it is usually convenient to go into momentum space. Using

$$(58) \quad K_{\mu\nu}(x-x') = \frac{1}{(2\pi)^4} \int dp \exp[ip(x-x')] K_{\mu\nu}(p)$$

and

$$(59) \quad S^{(1)}(x) = \frac{1}{(2\pi)^3} \int d^3p \exp[ipx] (i\gamma p - m) \delta(p^2 + m^2),$$

$$(60) \quad S_{R,A}(x) = \frac{1}{(2\pi)^4} \int d^4p \exp[ipx] (i\gamma p - m) \left\{ P \frac{1}{p^2 + m^2} \pm i\pi\varepsilon(p) \delta(p^2 + m^2) \right\},$$

where P denotes the principal value, one finds

$$(61) \quad K_{\mu\nu}(p) = \frac{1}{2} e^2 \frac{1}{(2\pi)^3} \int d^3p' d^3p'' \delta(p - p' + p'') \text{Sp} [\gamma_\mu (i\gamma p' - m) \gamma_\nu (i\gamma p'' - m)] \cdot \\ \cdot \left\{ \left[P \frac{1}{p'^2 + m^2} + i\pi\varepsilon(p') \delta(p'^2 + m^2) \right] \delta(p''^2 + m^2) + \right. \\ \left. + \left[P \frac{1}{p''^2 + m^2} - i\pi\varepsilon(p'') \delta(p''^2 + m^2) \right] \delta(p'^2 + m^2) \right\}.$$

The Lorentz covariant quantity $K_{\mu\nu}(p)$ must be expressed in terms of p_μ the only vector available, and clearly $K_{\mu\nu}(p)$ has the form

$$(62) \quad K_{\mu\nu}(p) = p_\mu p_\nu A(p) + \delta_{\mu\nu} p^2 B(p).$$

Since $\langle 0 | j_\mu(x) | 0 \rangle$ is a gauge invariant quantity; we must have

$$(63) \quad \int d^4x' K_{\mu\nu}(x - x') \frac{\partial}{\partial x'_\nu} A(x') = 0,$$

i.e.

$$(64) \quad p_\mu K_{\mu\nu}(p) = 0$$

so that

$$(65) \quad A(p) = -B(p)$$

and

$$(66) \quad K_{\mu\nu}(p) = (p_\mu p_\nu - \delta_{\mu\nu} p^2) G(p).$$

The problem thus reduces to the computation of $G(p)$. From (61) it is clear that

$$(67) \quad \left\{ \begin{array}{l} G(p) = -\frac{1}{3p^2} K_{\mu\mu}(p), \\ \text{and} \\ \text{Im } G(p) = -\text{Im } G(-p) \\ \text{Re } G(p) = +\text{Re } G(-p) \end{array} \right.$$

and we therefore write on grounds of invariance

$$(68) \quad \text{Im } G(p) = \pi \varepsilon(p) \Pi(p^2),$$

$$(69) \quad \text{Re } G(p) = \bar{\Pi}(p^2).$$

We note that to give an invariant meaning to (68), we should expect $\Pi(p^2) = 0$ for $p^2 > 0$. This will actually turn out to be the case. To simplify calculations we remark that on grounds of causality the kernel $K_{\mu\nu}(x - x')$ should vanish for $(x_0 - x'_0) < 0$. This is also evident from (56) since $S_R(x - x')$ and $S_A(x' - x)$ both contain retardation factors $\theta(x_0 - x'_0)$. Because of this, however, the real and imaginary parts of $G(p)$ are connected by a Hilbert transform, and actually

$$(70) \quad \bar{\Pi}(p^2) = \frac{1}{i\pi} P \int_{-\infty}^{\infty} dp'_0 \frac{i\pi \varepsilon(p'_0) \Pi(\bar{p}'^2 - p_0'^2)}{p'_0 - p_0}.$$

This relation may be transformed into

$$(71) \quad \bar{\Pi}(p^2) = P \int_0^{\infty} da \frac{\Pi(-a)}{a + p^2}.$$

Such an expression presupposes the convergence of the integral, and this question will arise later. For the time being (71) is to be regarded as a formal expression only.

Thus the result of our calculation is that

$$(72) \quad \begin{aligned} \langle 0 | J_\mu(x) | 0 \rangle &= \frac{1}{(2\pi)^4} \int dp \exp[ipx] (p_\mu p_\nu - \delta_{\mu\nu} p^2) (\bar{\Pi}(p^2) + i\pi \varepsilon(p) \Pi(p^2)) A_\nu^{\text{ext}}(p) = \\ &= - \frac{1}{(2\pi)^4} \int dp \exp[ipx] (\bar{\Pi}(p^2) + i\pi \varepsilon(p) \Pi(p^2)) J_\nu^{\text{ext}}(p). \end{aligned}$$

In order to compare the results of the calculation with experiment we must give some thought to what can actually be measured, and it is clear in the present instance that the observed current is a sum of the induced and the applied currents

$$J_\mu^{\text{obs.}}(x) = J_\mu^{\text{ext}}(x) + \langle 0 | J_\mu(x) | 0 \rangle$$

and we must make a convention to decide how the applied current is to be defined. We define it by stating that an external field constant in space and

time induces no current. This corresponds to writing

$$(73) \quad J_{\mu}^{\text{obs}}(x) = \frac{1}{(2\pi)^4} \int dp \exp[ipx] (1 + \bar{\Pi}(0) - \bar{\Pi}(p^2) - i\pi\varepsilon(p)\Pi(p^2)) J_{\mu}^{\text{ext}}(p),$$

which exhibits the property that the induced current vanishes in the long wave-length limit, or that the dielectric constant of the vacuum

$$(74) \quad \varepsilon(p) = 1 + \bar{\Pi}(0) - \bar{\Pi}(p^2) - i\pi\varepsilon(p)\Pi(p^2)$$

becomes unity in that limit.

The evaluation of $\text{Im } G(p)$ is straightforward, and the result is

$$(75) \quad \Pi(p^2) = \frac{e^2}{12\pi^2} \left(1 - \frac{2m^2}{p^2}\right) \sqrt{1 + \frac{4m^2}{p^2}} \theta(-p^2 - 4m^2),$$

The function $\Pi(p^2)$ is seen to vanish not only for $-p^2 < 0$ as was expected on grounds of invariance but also for $-p^2 < 4m^2$. When $-p^2 > 4m^2$, $\Pi(p^2)$ is no longer zero, which means that the dielectric constant has an imaginary part. It is well known that this represents a possibility of energy absorption, and here it means that the vacuum can take energy from the source. It does so by the process of pair creation, and the threshold for such a process appears in the factor $\theta(-p^2 - 4m^2)$. Since $\Pi(-a)$ approaches a constant as $a \rightarrow \infty$ the function $\bar{\Pi}(p^2)$ as defined by (71) is infinite. This fact however turns out to be quite irrelevant because an examination of the physics of the problem showed us that only

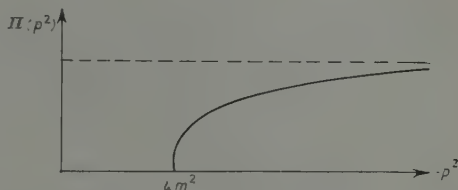


Fig. 3. - Plot of $\Pi(p^2)$ as a function of $-p^2$.

$$(76) \quad \bar{\Pi}(p^2) - \bar{\Pi}(0) = -p^2 \int da \frac{\Pi(-a)}{a(a + p^2)}$$

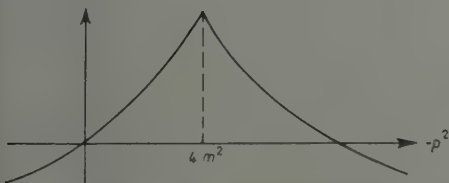


Fig. 4. - Plot of the function $\bar{\Pi}(p^2) - \bar{\Pi}(0)$.

is observable, and this quantity is seen to be finite.

The real part of the dielectric constant represents vacuum fluctuations brought about by the creation and subsequent annihilation of virtual pairs, for example.

The effects calculated above have actually been observed. The effects represented by $\Pi(p^2)$ appear in the $0 \rightarrow 0$ transition in ^{16}O which proceeds by pair emission. The vacuum fluctuations $\bar{\Pi}(p^2) - \bar{\Pi}(0)$ contribute -27 Me to the Lamb Shift, and are clearly present.

One can carry out calculations to any order in perturbation theory using methods similar to those used above. In manipulating singular functions one must however be very careful always to deal with well defined quantities. As an illustration of a pitfall sometimes encountered in this field, let us calculate the matrix element $\langle 0 | A_\mu(x) | k \rangle$ where $|k\rangle$ represents a state of a single incoming photon. From (53)

$$(77) \quad \langle 0 | A_\mu(x) | k \rangle = \langle 0 | A_\mu^{\text{in}}(x) | k \rangle + \int dx' D^{\text{ret}}(x - x') \langle 0 | j_\mu(x') | k \rangle$$

and

$$(78) \quad \langle 0 | J_\mu(x) | k \rangle = K_{\mu\nu}(k) \langle 0 | A_\nu^{\text{in}}(x) | k \rangle,$$

where

$$(79) \quad K_{\mu\nu}(k) = (k_\mu k_\nu - \delta_{\mu\nu} k^2) (\bar{\Pi}(k^2) - \bar{\Pi}(0) + i\pi\epsilon(k) \Pi(k^2)).$$

For the physical photon $k^2 = 0$ so that

$$(80) \quad \begin{cases} \Pi(k^2)|_{k^2=0} = 0 \\ \bar{\Pi}(k^2) - \bar{\Pi}(0)|_{k^2=0} = 0 \end{cases}$$

and therefore

$$(81) \quad \langle 0 | J_\mu(x) | k \rangle = 0$$

so that

$$(82) \quad \langle 0 | A_\mu(x) | k \rangle = \langle 0 | A_\mu^{\text{in}}(x) | k \rangle.$$

A glance at eq. (77) shows however that the matrix element $\langle 0 | J_\mu(x') | k \rangle$ which vanishes as $k^2 \rightarrow 0$ is multiplied by the singular function $D_R(x - x')$ whose Fourier transform behaves like $1/k^2$. If one makes all integrals well defined by introducing an «adiabatic switching off the charge» by writing

$$(83) \quad e A_\nu^{\text{in}}(x) \rightarrow e \exp[-\alpha |x'_0|] A_\nu^{\text{in}}(x')$$

then as $k^2 \rightarrow 0$

$$\underline{k}^2 \rightarrow \bar{k}^2 - (k_0 + i\alpha)^2 \rightarrow -2i\alpha k_0.$$

Thus

$$(84) \quad K_{\mu\nu}(k) = (k_\mu k_\nu + 2i\alpha k_0 \delta_{\mu\nu})(-2i\alpha k_0 \bar{\Pi}'(0)) .$$

After some simple calculations one finds that the correct result is

$$(85) \quad \langle 0 | A_\mu(x) | k \rangle = \left(\delta_{\mu\nu} + k_\mu k_\nu \frac{\bar{\Pi}'(0)}{2} \right) \langle 0 | A_\nu^{\text{in}}(x) | k \rangle .$$

The quantity $\bar{\Pi}'(0)$ turns out to be $(1/137)(1/30\pi)$.

The possibility of such a mistake is a warning to be very careful in replacing quantities like

$$\int \square D^{\text{ret}}(x - x') A_\mu^{\text{in}}(x') dx' ,$$

by others like

$$\int D^{\text{ret}}(x - x') \square A_\mu^{\text{in}}(x') dx' .$$

These differ only in surface terms, *i.e.* by terms which depend on the asymptotic behavior of the field quantities, and great care must therefore be exercised in specifying such asymptotic conditions for the operators.

4. - The Lee model.

In order to obtain a clearer insight into the treatment of current field theory, and in particular the problem connected with renormalization, it is constructive to study a particularly simple model in which many calculations can be done exactly and the meaning of renormalization can be clearly stated.

In this model there are three particles, θ , N , V , which can be transmuted into each other thus

$$(86) \quad V \leftrightarrow N + \theta ,$$

but not thus

$$(87) \quad N \leftrightarrow V + \theta .$$

The Hamiltonian corresponding to such a system is

$$(88) \quad \left\{ \begin{array}{l} H = H_0 + H_1 , \\ H_0 = m_V \sum_{\vec{p}} \psi_V^*(\vec{p}) \psi_V(\vec{p}) + m_N \sum_{\vec{p}} \psi_N^*(\vec{p}) \psi_N(\vec{p}) + \sum_{\vec{k}} \omega(k) a^\dagger(k) a(k) , \\ H_1 = - \frac{g_0}{\sqrt{V}} \sum_{\vec{p}, \vec{p}', \vec{k}} \left\{ \psi_V^*(\vec{p}) \psi_N(\vec{p}') a(\vec{k}) \frac{f(\omega)}{\sqrt{2\omega}} + \text{h. c.} \right\} . \end{array} \right.$$

Here $f(\omega)$ is a cut-off function, $\omega = \sqrt{\mu^2 + \bar{k}^2}$

$$(89) \quad \left\{ \begin{array}{l} [a(\bar{k}), a^+(\bar{k}')] = \delta_{\bar{k}\bar{k}'} , \\ \{\psi_V(\bar{p}), \psi_V^*(\bar{p}')\} = \{\psi_N(\bar{p}), \psi_N^*(\bar{p}')\} = \delta_{\bar{p}\bar{p}'}, \text{ etc.} \end{array} \right.$$

Because of the stringent selection rule (87) there are two additional constants of the motion in the theory, and these make the problem soluble in a simple form. We shall work with TD states because of the simplicity of the problem, and define the general TD state by $|n_V, n_N, n_{\bar{k}}\rangle$ the labels referring to the number of V , N and θ particles present. The momentum label is only on the θ particle because the others are assumed to be fixed in space.

Let us now look at some simple physical states: it is easy to see that

$$(90) \quad \left\{ \begin{array}{l} |0\rangle = |0, 0, 0\rangle, \\ |\theta\rangle = |0, 0, 1_{\bar{k}}\rangle, \\ |N\rangle = |0, 1, 0\rangle, \end{array} \right.$$

$$(91) \quad |V\rangle = N\{|1, 0, 0\rangle + \sum_{\bar{k}} \Phi(\bar{k})|0, 1, 1_{\bar{k}}\rangle\}.$$

The last state, because of the selection rule has been expressed as a superposition of a bare V particle and a superposition of $N+\theta$ only. The quantity N is a normalization constant. Substituting (91) into

$$(92) \quad H|V\rangle = E_V|V\rangle$$

we obtain

$$(93) \quad \Phi(\bar{k}) = \frac{g_0}{\sqrt{2\omega V}} f(\omega) \frac{1}{m_N + \omega - E_V}.$$

We shall assume that the physical V particle is stable so that

$$E_V < m_N + \mu.$$

The energy of the physical V particle E_V is a solution of the eigenvalue equation

$$(94) \quad F(E_V - m_N) = m_V - E_V,$$

where

$$(95) \quad F(z) = \gamma_0 \int_{\mu}^{\infty} \frac{k f^2(\omega) d\omega}{\omega - z}$$

and

$$(96) \quad \gamma_0 = g_0^2/4\pi.$$

Furthermore

$$(97) \quad \frac{1}{N^2} = 1 + F'(E_V - m_N).$$

The next simple state we can solve for exactly is $|N+\theta\rangle$. Writing

$$(98) \quad |N+\theta\rangle = |0, 1, 1_{\vec{k}}\rangle + \sum_{\vec{k}'} \alpha(\vec{k}, \vec{k}') |0, 1, 1_{\vec{k}}\rangle + \beta(\vec{k}) |1, 0, 0\rangle,$$

we find that

$$(99) \quad \alpha^{\text{in, out}}(\vec{k}, \vec{k}') = \frac{g_0}{\sqrt{2\omega V}} f(\omega') \frac{\beta^{\text{in, out}}(\vec{k})}{\omega' - \omega \mp i\varepsilon},$$

$$(100) \quad \beta^{\text{in, out}}(\vec{k}) = \frac{g_0}{\sqrt{2\omega V}} f(\omega) \frac{1}{m_V - m_N - \omega - F(\omega \pm i\varepsilon)}.$$

From this we may calculate the S -matrix for $N+\theta$ scattering

$$(101) \quad \langle N'\theta' | S | N\theta \rangle = \langle N'\theta', \text{out} | N\theta, \text{in} \rangle = \delta_{\vec{k}\vec{k}'} + \frac{2\pi i g_0}{\sqrt{2\omega V}} f(\omega) \beta(k) \delta(\omega - \omega').$$

Since everything is spherically symmetric in (101) we have only S -wave scattering, and the phase shift is given by

$$(102) \quad \text{tg } \delta = -\frac{1}{2} F(\omega) / \chi(\omega).$$

$$(103) \quad \begin{cases} F(\omega) = 2\pi\gamma_0 f^2(\omega)k & (\text{essentially determined by } \text{Im } F(\omega + i\varepsilon)) \\ \chi(\omega) = m_V - m_N - \omega - \gamma_0 P \int_{\mu}^{\infty} d\omega' \frac{k' f^2(\omega')}{\omega' - \omega} \\ & (\text{essentially determined by } \text{Re } F(\omega + i\varepsilon)). \end{cases}$$

If we now take the model seriously and wish to put it in a form to be compared with experiment, we must look at the physical significance of the parameters of the theory, μ , m_N , m_V , γ_0 . For simplicity we assume that the cut-off function $f(\omega)$ is given. The parameters μ and m_N are physically measurable, because the bare and physical N and θ particles are identical. This is not true of m_V and we should really express this parameter in terms of the physically measurable energy of the physical V particle E_V . When this

is done using (94), we find for the scattering phase shift

$$(104) \quad \operatorname{tg} \delta = - \frac{\pi k f^2(\omega)}{(E_V - m_N - \omega) \left(\frac{1}{\gamma_0} + P \int_{\mu}^{\infty} \frac{k' f^2(\omega') d\omega'}{(\omega' - \omega)(\omega' - E_V + m_N)} \right)}.$$

Note here that this re-expression of the phase shift in terms of the physical quantity E_V has incidentally led to an extra denominator in the integral occurring in $\operatorname{tg} \delta$. To measure the remaining parameter γ_0 we do a hypothetical experiment, and determine, say, the scattering length a ,

$$a = \left(\frac{\operatorname{tg} \delta(k)}{k} \right)_{k=0}.$$

We then express γ_0 in terms of that quantity and find that

$$(105) \quad \frac{1}{\gamma_0} = - \frac{\pi f^2(\omega)}{a(E_V - m_N - \mu)} - P \int_{\mu}^{\infty} \frac{k' f^2(\omega') d\omega'}{(\omega' - \mu)(\omega' - E_V + m_N)}$$

and

$$(106) \quad \frac{\operatorname{tg} \delta}{k} = \pi a f^2(\omega) \frac{E_V - m_N - \mu}{E_V - m_N - \omega} \cdot \frac{1}{\pi f^2(\mu) + a(\omega - \mu)(\mu + m_N - E_V) P \int_{\mu}^{\infty} \frac{k' f^2(\omega') d\omega'}{(\omega' - \mu)(\omega' - \omega)(\omega' - E_V + m_N)}}.$$

Note that the integral in the denominator is convergent even in the point source limit $f(\omega) \rightarrow 1$.

Usually the calculations are not expressed in terms of the results of an experiment, but rather in terms of a «renormalized coupling constant» which is defined by demanding that the scattering length be exactly equal to the lowest order perturbation result of the theory in terms of the renormalized coupling constant. This amounts to saying that

$$(107) \quad \left[\frac{\operatorname{tg} \delta(k)}{k} \right]_{k=0} = \gamma_0 \frac{f^2(\mu)}{m_N + \mu - E_V} + O(\gamma_0^2) \equiv \gamma \frac{f^2(\mu)}{m_N + \mu - E_V},$$

so that

$$(108) \quad a \equiv \gamma \frac{f^2(\mu)}{m_N + \mu - E_V}.$$

We then get

$$(109) \quad \frac{\operatorname{tg} \delta}{k} = \frac{\gamma \pi f^2(\omega)}{\omega + m_N - E_V} \cdot \frac{1}{1 + (\omega - \mu) \gamma P \int_{\mu}^{\infty} \frac{k' f^2(\omega') d\omega'}{(\omega' - \mu)(\omega' - E_V + m_N)(\omega' - \omega)}}$$

The convention of using a threshold measurement to define this coupling constant is quite arbitrary. It is sometimes mathematically convenient to define the coupling constant in terms of the scattering at the unphysical energy $\omega = 0$, i.e. in terms of a result extrapolated from the physical region.

It is to be noted that the relation between γ and γ_0

$$(110) \quad \gamma = \frac{\gamma_0}{1 + \gamma_0 \int_{\mu}^{\infty} d\omega' \frac{k' f^2(\omega')}{(\omega' - \mu)(\omega' - E_V + m_N)}}$$

shows that γ is not a completely adjustable parameter. As (110) shows this is not the case, and in the point source limit $f(\omega) = 1$, $\gamma = 0$ unless $\gamma_0 < 0$, in which case however g_0 is imaginary the Hamiltonian is no longer hermitean, and many strange features may be expected.

It is a remarkable fact that after performing the «mass renormalization» and the «coupling constant renormalization» the physically measurable quantities turn out to be finite in this theory, and incidentally also in perturbation expansions in quantum electrodynamics. We must stress however that in all conventional theories in which the masses and coupling strengths are experimentally determined numbers, this renormalization procedure, in which all theoretical parameters are expressed in terms of the experimental numbers, is an essential part of the theory, without which contact with experiments cannot be made.

With $E_V = m_N$ for simplicity

$$(111) \quad \gamma = \gamma_0 / \left(1 + \gamma_0 \int_{\mu}^{\infty} \frac{k f^2(\omega) d\omega}{\omega^2} \right); \quad g = g_0 \sqrt{\gamma / \gamma_0}$$

and comparison with earliest results shows that

$$(112) \quad \frac{\gamma}{\gamma_0} = N^2 = 1 - \gamma \int_{\mu}^{\infty} \frac{k f^2(\omega)}{\omega^2} d\omega.$$

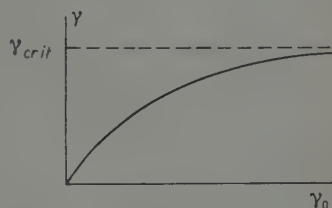


Fig. 5. — Plot of γ vs. γ_0 .

In terms of the renormalized g ,

$$(113) \quad \alpha(k, k') = \frac{gf(\omega')}{\sqrt{2\omega'V}} \frac{\beta_r(k)}{\omega' - \omega - i\varepsilon}$$

and

$$(114) \quad \beta_r(\bar{k}) = N\beta(\bar{k}) = \frac{gf(\omega)}{\sqrt{2\omega V}} \frac{1}{h(\omega + i\varepsilon)},$$

where

$$(115) \quad h(z) = z \left(1 + \gamma z \int_{\mu}^{\infty} \frac{k f^2(\omega) d\omega}{\omega^2(\omega - z)} \right).$$

Let us now look at some matrix elements. Clearly

$$(116) \quad \langle 0 | \psi_N | N \rangle = 1,$$

also

$$(117) \quad \langle \theta_{\bar{k}} | \psi_N | N, \theta \rangle = \delta_{\bar{k}\bar{k}'} + \alpha(\bar{k}, \bar{k}').$$

In general all matrix elements involving only N and θ particles are finite in the point source limit. However

$$(118) \quad \langle 0 | \psi_V | V \rangle = N$$

and

$$(119) \quad \langle 0 | \psi_V | N, \theta \rangle = N\beta_V(\bar{k}).$$

If we define

$$(120) \quad \psi'_V = \frac{1}{N} \psi_V,$$

then

$$(121) \quad \langle 0 | \psi'_V | V \rangle = 1$$

and

$$(122) \quad \langle 0 | \psi'_V | N, \theta \rangle = -\frac{gf(\omega)}{\sqrt{2\omega V}} \frac{1}{h(\omega + i\varepsilon)}.$$

This redefinition of the operator ψ_V (we shall hereafter drop the prime and always understand $1/N$ times the original operator) is usually (and slightly confusingly) called « wave function renormalization ».

In this model only the V particle operator required renormalization. In general all fields have to be renormalized.

It is of interest to examine the high energy behavior of the matrix elements. With a reasonably smooth cut-off function

$$(123) \quad \lim_{\omega \rightarrow \infty} h(\omega + i\epsilon) = \omega \left[1 - \gamma \int_{\mu}^{\infty} \frac{k' f^2(\omega') d\omega'}{\omega'^2 (1 - \omega'/\omega)} \right] = \omega \left[1 - \gamma \int_{\mu}^{\infty} \frac{k' f^2(\omega')}{\omega'^2} d\omega' \right] = \omega N^2.$$

Now we notice that the Born approximation of $\langle 0 | \psi_V | N, \theta \rangle$ is

$$(124) \quad \langle 0 | \psi_V^{B.A.} | N, \theta \rangle = - \frac{gf(\omega)}{\sqrt{2\omega V}} \frac{1}{\omega},$$

so that

$$(125) \quad \lim_{\omega \rightarrow \infty} \langle 0 | \psi_V | N, \theta \rangle = \frac{1}{N^2} \langle 0 | \psi_V^{B.A.} | N, \theta \rangle.$$

Thus the high energy limit of the matrix element reduces to lowest order perturbation theory. Intuitively we may interpret this result by the statement that at very high energies there is very little interaction, that the colliding particles penetrate the virtual clouds surrounding the particles, and that therefore the result of the interaction appears only in its lowest order perturbation form. The unaccounted for factor N^2 acts to change g into $g/N = g_0$ and ψ_V to $\psi_V^{unr.} = N\psi_V$, so that the result of our calculation is that at high energies the matrix elements are given by the Born approximation for the *unrenormalized theory*. This result is exact in the Lee model. It has been shown to be true for a particular matrix element in quantum electrodynamics, and its physical reasonableness may make one hope that it is generally true.

On quite general grounds one may expect the result that $0 \leq N^2 \leq 1$. To see this we consider the anticommutation relation (89) for the renormalized ψ_V . Then

$$(126) \quad \frac{1}{N^2} = \langle 0 | \{ \psi_V^*(\vec{p}), \psi_V(\vec{p}) \} | 0 \rangle = \sum_{|z\rangle} \{ \langle 0 | \psi_V^* | z \rangle \langle z | \psi_V | 0 \rangle + \langle 0 | \psi_V | z \rangle \langle z | \psi_V^* | 0 \rangle \}.$$

The first term vanishes. In the second term there are contributions from bare V states and from the $|N\theta\rangle$ scattering states. Thus

$$(127) \quad \frac{1}{N^2} = 1 + \sum |\beta(\vec{k})|^2,$$

which leads to the result. In (112) we had

$$(128) \quad N^2 = 1 - \gamma/\gamma_{crit},$$

so that in the case when the coupling constant is below its permissible maximum nothing unexpected occurs. If one lets γ become too large, pathological features appear in the theory, and one may ask whether these are common to all field theories.

An examination of this question shows that when all fields are renormalized there is no conclusive reason why such troubles should occur.

5. — Remarks on quantum electrodynamics.

Just as in the Lee model, the unrenormalized fields $A_\mu^{\text{u.r.}}(x)$ and $\psi^{\text{u.r.}}(x)$ and the charge, or coupling constant e_0 are renormalized by multiplicative factors, and the renormalized equations of motion are

$$(129) \quad \left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m \right) \psi(x) = ie\gamma_\mu A_\mu(x) \psi(x) + \delta m \psi(x)$$

and

$$(130) \quad \square A_\mu(x) = -\frac{ieN^2}{2} [\bar{\psi}(x), \gamma_\mu \psi(x)] + L \left[\square A_\mu(x) - \frac{\partial^2}{\partial x_\mu \partial x_\nu} A_\nu(x) \right].$$

In the first equation a mass-renormalization counter term was explicitly added. This was done to make the task of expressing the bare mass in terms of the physically measurable mass easier in practice, and m therefore is the physical mass. In the second equation the last term on the r.h.s. is a gauge term which again makes some of the technical manipulations easier to carry out. The renormalization constants L , N and δm , where

$$(131) \quad e = \sqrt{1-L} e_0; \quad A_\mu(x) = \frac{1}{\sqrt{1-L}} A_\mu^{\text{u.r.}}(x); \quad \psi(x) = \frac{1}{N} \psi^{\text{u.r.}}(x),$$

can be expressed in terms of matrix elements of the operators.

Thus for example, as was first shown by UMEZAWA and KAMEFUCHI

$$(132) \quad \frac{1}{1-L} = 1 + \int da \frac{\Pi(-a)}{a} = 1 + \Pi(0),$$

where

$$(133) \quad \Pi(p^2) = -\frac{V}{3\mu^2} \sum_{p(z)=p} \langle 0 | j_\mu | z \rangle \langle z | j_\mu | 0 \rangle.$$

The function $\Pi(p^2)$ enters into the vacuum polarization. We dealt with it earlier in the course of a perturbation treatment of the external field and

the radiation field. Actually the formula

$$(134) \quad j_{\mu}^{\text{obs}}(x) = \frac{1}{(2\pi)^4} \int dp \exp[ipx] j_{\mu}^{\text{ext}}(p) [1 - \bar{\Pi}(p^2) + \Pi(0) - i\pi\varepsilon(p)\Pi(p^2)],$$

where $\bar{\Pi}(p^2)$ is defined by (71) can be derived by using perturbation theory in the external field only. The function $\Pi(p^2)$ also enters into the calculation of the vacuum fluctuation of the average current

$$(135) \quad j_{\mu}(R) = \int dx j_{\mu}(x) f_R(x)$$

(here $f_R(x)$ is a «test function» which vanishes outside a certain space time region R). One finds that

$$(136) \quad \langle 0 | j_{\mu}(R) j_{\nu}(R) | 0 \rangle = \int dp (p_{\mu} p_{\nu} - \delta_{\mu\nu} p^2) \Pi(p^2) |G_R(p)|^2,$$

where $G_R(p)$ is the Fourier transform of the (arbitrary) test function. If we take $\mu = \nu = 1$, say, and note that $G_R(p)$ is arbitrary, we see that $\Pi(p^2)$ must be positive. Actually this can be proved in a very general way. As in the perturbation example discussed earlier, the quantity

$$(137) \quad \bar{\Pi}(p^2) - \bar{\Pi}(0) = -p^2 P \int_0^{\infty} \frac{da \Pi(-a)}{a(a+p^2)},$$

appears in an observable, and ought to be finite if the theory makes sense. It is clear therefore, that to discuss the finiteness of the renormalization constants and of the observable quantities, we must know the high energy behavior of $\Pi(p^2)$. The fact that $\Pi(p^2)$ is a sum of positive terms (cf. (133)) is of great help here since we may hope to obtain some information about the high energy behavior of $\Pi(p^2)$ by considering a partial sum of the terms in (133). Thus

$$(138) \quad \Pi(p^2) > \frac{V}{-3p^2} \sum_{q+q'=p} \langle 0 | j_{\mu} | q, q' \rangle \langle q, q' | j_{\mu} | 0 \rangle,$$

where the states $|q, q'\rangle$ are states with one incoming electron-positron pair. One may show that if all the renormalization constants are finite

$$(139) \quad \lim_{-(q+q')^2 \rightarrow \infty} \langle 0 | j_{\mu} | q, q' \rangle = \frac{N^2}{1-L} \langle 0 | j_{\mu}^B | q, q' \rangle,$$

where the matrix element on the r.h.s. is the Born approximation for the unrenormalized theory, a result quite similar to that found in the Lee model (*). From this it is easy to show that

$$(140) \quad \Pi(p^2) \geq \frac{N^4}{(1-L)^2} \frac{e_0^2}{12\pi^2},$$

which leads to an inconsistency unless at least one of the renormalization constants is infinite.

A much more important question is whether the observable quantities in the theory, *i.e.* expressions of the type

$$(141) \quad \bar{\Pi}(p^2) - \bar{\Pi}(0) = - \int_0^\infty \frac{da \Pi(-a)}{a(1 + (a/p^2))},$$

are finite, even though the renormalization constants be infinite. In this case we can no longer say that the bare fields and charges exist, but that at higher energies « effective » fields and charges appear, which for example have the form

$$(142) \quad e^2(p^2) = e^2 [(1 - \bar{\Pi}(p^2) + \bar{\Pi}(0))^2 + \pi^2 \Pi^2(p^2)]^{\frac{1}{2}}.$$

Note that (142) approaches $1/(1-L)$ if $\bar{\Pi}(0)$ is finite. Similarly the factor N^2 is replaced by a function $N^2(p^2)$ which approaches zero as p^2 becomes very large. Thus (139) becomes

$$(143) \quad \lim_{-p^2 = -(q+q')^2 \rightarrow \infty} \langle 0 | j_\mu | q, q' \rangle = \sqrt{\frac{e^2(p^2)}{e^2}} N(p^2) \langle 0 | j_\mu^B | q, q' \rangle.$$

The convergence of the integral $\int_0^\infty da \Pi(-a)/a^2$ implies that

$$(144) \quad \lim_{p^2 \rightarrow \infty} \frac{e^2(p^2)}{p^2} = 0$$

and one may similarly show that $N^2(p^2)$ does not decrease faster than some power of p^2 . To go on from here we make the *assumption* that the high energy

(*) As we have already shown, the renormalization constant $1/(1-L)$ is infinite in perturbation theory. Nevertheless it could be that the renormalization constant calculated without the use of perturbation theory could be finite. For example if $\Pi(0) \sim 1/(1+e^2(\log 0)^2)$ such a situation would arise.

behavior of the matrix elements at very high energies (for all the particles involved) is determined by the Born approximation for the theory characterized by the appropriate effective charges and fields. Thus if $|z\rangle$ represents the state with ν incident photons k_1, \dots, k_ν then

$$(145) \quad \langle 0 | j_\mu | z \rangle \simeq \left(\frac{e^2(p^2)}{e^2} \right)^{\frac{\nu}{2}} N(p^2) \langle 0 | j_\mu^B | z \rangle.$$

Again a lower limit to $\Pi(p^2)$ can be obtained by summing over all states containing photons only, and in particular only those photons having high enough energies so that they interact in an effectively weak coupling manner. Thus if the energy above which the asymptotic form holds is E_0 then we include $\bar{N} \ll \sqrt{-j^i/E_0^2}$ photons. We thus find

$$(146) \quad \Pi(p^2) > N^2(p^2) \sum_{\nu=1,2,\dots,\bar{N}} \left(\frac{e^2(p^2)}{e^2} \right)^{\frac{\nu}{2}} \frac{V}{3p^2} \sum_{\Sigma k_i=p} \langle 0 | j_\mu^B | k_1 \dots k_\nu \rangle \langle k_1 \dots k_\nu | j_\mu^B | 0 \rangle.$$

The r.h.s. of (146) is proportional to the transition probabilities for the emission of ν photons by a weak external field, and the result is a Poisson distribution with a mean number of photons of the order of $[\log -p^2/m^2]^2$. Thus on the basis of our conjecture about the behavior of the matrix elements at high energies we can evaluate (146) and find that

$$(147) \quad \Pi(p^2) > CN^2(p^2) \sum_{\nu=1,\dots,\bar{N}} \frac{(e^2(p^2))^\nu}{\nu!} \left[\log \frac{-p^2}{m^2} \right]^{2\nu},$$

i.e. in the high energy limit

$$(148) \quad \Pi(p^2) > N^4(p^2) \exp [\alpha(p^2) [\log (-p^2/m^2)]^2]$$

so that $\Pi(a)$ increases faster than any power of a .

We have therefore shown that if the conjecture made above is correct, the theory does not give any finite answers, and because of the exponential behavior of $\Pi(a)$ no amount of renormalization can fix it up.

To show whether the conjecture is indeed correct in general, one is led to the study of

$$\langle 0 | j_\mu | k_1 \dots k_\nu \rangle = i^n \int dx_1 \dots dx_\nu \theta(x - x_1) \dots \theta(x^{\nu-1} - x^\nu) \cdot \\ \cdot \langle 0 | [j_{\mu_\nu}(x^\nu), [j_{\mu_{\nu-1}}(x^{\nu-1}), \dots, [j_{\mu_1}(x_1), j_\mu(x)] \dots]] | 0 \rangle A_{\mu_1}^{(0)}(x_1) \dots A_{\mu_\nu}^{(0)}(x_\nu) | k_1 \dots k_\nu \rangle,$$

which brings us to the theory of the general structure of vacuum expectation values of retarded (n -fold) commutators.

The complications encountered even in the case of the three-fold vacuum expectation value are so large that it is difficult to envisage an early proof or disproof of the conjecture.

In conclusion we may briefly speculate as to the situation in case the conjecture should turn out to be correct. The evaluation of (146) hinged on the fact that $\langle 0 | j_\mu^B | k_1 \dots k_r \rangle$ had the Poisson behavior, which is a classical limit effect, and which cannot be done away with. The only hope of modifying the conclusion in a simple way without affecting the excellent agreement with experiment in the low energy region is to change the positive definite character of $\Pi(p^2)$ at high energies. This might be expected to happen at energies of the order of mesonic or nucleonic masses, and it would involve the introduction of an indefinite metric. With such a modification of the theory, problems arise with the unitarity of the S -matrix (which in effect expresses the fact that all probabilities for observable events are positive and add up to one) and it is not clear at this stage whether the various proposals made can be made consistent with physical principles.

The Framework of Quantum Field Theory.

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1. - Introduction.

The purpose of these lectures is to give an account of a particular approach to Quantum Field Theory which has been used by several people in the past 5 years. It has been called the « axiomatic method » (a name which I dislike intensely) or « approach from basic principles » or « systematic analysis of the frame ». No spectacular developments have resulted from it so far but it furnishes us at least with some clear cut mathematical questions, some of which seem to be answerable within a finite amount of time.

The motivation for setting out on such a relatively uninspiring program may be understood from the following remarks. In the period between 1930 and 1948 most physicists were convinced that there was something basically wrong in the scheme of quantum field theory. Direct evidence for this was seen in the divergence of higher order terms in the perturbation expansion. The consequence of this feeling was a large number of different proposals for change: change in the formalism, change in the basic concepts, assumption of new physical principles. Then, for a few years, faith in the original scheme was restored by the success of renormalization which showed how to extract finite results from the higher order terms and, more important still, that the information thus obtained was physically relevant in Quantum Electrodynamics. However, the gloomy outlook returned again since the basic problem had remained unsolved. We do not know whether the set of rules which is called « renormalized field theory » actually defines a consistent theory nor have we learned to formulate a relativistic field theory without recourse to an extremely unclear terminology. Characteristically enough the inadequacy of the conventional theory as well as of all proposals for change lies not in the fact that

they were tested and found out to be wrong but in the fact that we do not have the strength to test them. I think this is a very important point to bear in mind if one wants to think about the time scale in which essential progress may be expected. It may very well be that we have to work patiently for many years just to get sufficient control of the mathematics of an infinite system of operators before we can discuss fruitfully the merits of new physical ideas. (Of course these remarks refer only to the treatment of the dynamical problem, not to the classification scheme of elementary particles.)

This is now exactly the frame of mind which leads to the systematic analysis approach.

We start with the question:

How far do the physical principles as we know (or understand) them today go in fixing the mathematical frame of the theory? Of course, since we are only interested in a special sector of physics, namely the description of interaction processes between small particles, we need to be concerned only with those principles which we believe to be relevant for this particular sector. Already at this point we have to make the decision, whether the general theory of relativity is relevant to our purposes or not. We assume here that the general theory of relativity may be ignored, but it is by no means clear whether this decision is a wise one. The principles which we do want to incorporate then are the following three:

- I) The basic mathematics of Quantum Physics.
- II) The invariance of the theory under inhomogeneous Lorentz transformations.
- III) The fact that all the states of the system we want to describe correspond to collision processes between a number of particles. This particle number may be arbitrarily large but never infinite.

The consequences of the first two postulates (quantum physics plus Lorentz invariance) have been formulated in their most general mathematical form by E. P. WIGNER partly together with V. BARGMANN and A. S. WIGHTMAN and have been reported in great detail in this course in the lectures of L. MICHEL and A. S. WIGHTMAN (which will be quoted as M-W here). I can therefore restrict myself in this respect to a very brief summary. The third postulate serves to select that particular type of representation of the inhomogeneous Lorentz group which will be of interest to us in quantum field theory. In other words, it will make a statement about the required type of spectrum for the infinitesimal operators of the group, among which are the Hamiltonian, the Momentum, the Angular Momenta. I shall therefore call the consequences of principle III the «spectral conditions». Their formulation and condensation into a mathematical formalism will be the first part of this

lecture. At this stage, in spite of using terms like « incoming field » our frame has no connection as yet to a quantum field theory. It is essentially a general S -matrix frame. The question of how to pose the dynamical problem, which eventually will determine the S -matrix, is not yet discussed.

We establish the connection with field theory by assuming that the basic quantities in which all the previously defined notions have to be expressed are operator fields with a certain set of properties which will be discussed in the second part of the lecture.

Finally, we will discuss the connection between the field quantities and the S -matrix, the physical basis for assuming certain properties for the field operators and related questions.

2. – Spectral conditions.

To summarize briefly the consequences of the first two requirements (Quantum Physics plus Special Relativity) for the mathematical formalism:

We are dealing with a Hilbert space \mathcal{H} and a representation of the inhomogeneous Lorentz Group by unitary operators $U(a, A)$ in this space.

We will now make use of the particle postulates III to make statements about the type of representation $U(a, A)$ with which we have to deal.

First, there should be 1-particle-states. Those states correspond (according to the analysis reported by M.-W.) to irreducible representations of the inhomogeneous Lorentz group of type (s, m) : (s – spin, m = mass value). For simplicity we discuss here only the case of spinless particles ($s = 0$) with non-vanishing rest mass ($m \neq 0$). No distinction between « elementary » and « composite » particles is made.

Postulate 1. There shall be certain discrete, invariant subspaces in \mathcal{H} called $\mathcal{H}^{(j)}$ which transform under irreducible representations of the Lorentz group. These representations shall be of type $(0, m_j)$ (*).

2'1. Description of the states from $\mathcal{H}^{(j)}$. – A general state of one particle is characterized (see M.-W.) by a wave function $\tilde{f}(p)$, where p stands for the 4-vector \mathbf{p} , p_0 . We adopt the convention of writing the Lorentz scalar product between two 4-vectors as $xp = \mathbf{x} \cdot \mathbf{p} - x_0 p_0$. The wave function is defined only on the hyperboloid $p^2 = -m^2$, $p_0 > 0$, but, for convenience, we will sometimes assume it to be extended in a continuous fashion into a small neighborhood of this hyperboloid. The scalar product in Hilbert space between

(*) Of course, since every representation of the Lorentz group is decomposable into a direct integral of irreducible representation, the emphasis lies on the fact that $\mathcal{H}^{(j)}$ is a discrete subspace, i.e. that its vectors are normalizable states of the total space \mathcal{H} .

two states with wave functions \tilde{f}_1 and \tilde{f}_2 is

$$(1) \quad (\tilde{f}_1, \tilde{f}_2) = \int \tilde{f}_1^*(\mathbf{p}, E_p) \tilde{f}_2(\mathbf{p}, E_p) \left(\frac{d\mathbf{p}}{2E_p} \right) = \\ = \int \tilde{f}_1^*(p) \tilde{f}_2(p) \delta(p^2 + m^2) d^4p \quad \text{with} \quad E_p = \sqrt{\mathbf{p}^2 + m^2}.$$

The transformation properties are $(U(a, \Lambda)f = f')$

$$(2) \quad \left\{ \begin{array}{l} \text{Translation by 4-vector } a: \\ \tilde{f}'(p) = \exp[-ipa] \tilde{f}(p) \\ \text{Homogeneous L.T. } \Lambda: \\ \tilde{f}'(p) = \tilde{f}(\Lambda^{-1}p). \end{array} \right.$$

Sometimes it is useful to go over to the wave functions in position space

$$(3) \quad f(x) = (2\pi)^{-\frac{3}{2}} \int \tilde{f}(\mathbf{p}, E_p) \exp[i\mathbf{p} \cdot \mathbf{x} - E_p x_0] \frac{d\mathbf{p}}{2E_p},$$

$f(x)$ is obviously a solution of the Klein-Gordon equation

$$(4) \quad (\square - m^2)f(x) = 0.$$

We shall also sometimes use the Newton-Wigner wave functions in momentum or position space

$$(5) \quad \tilde{\varphi}(\mathbf{p}) = \frac{\tilde{f}(\mathbf{p}, E_p)}{\sqrt{2E_p}}; \quad \varphi(x) = (2\pi)^{-\frac{3}{2}} \int \tilde{\varphi}(\mathbf{p}) \exp[i\mathbf{p} \cdot \mathbf{x} - E_p t] d\mathbf{p}.$$

The advantage of these wave functions is that the scalar product takes a simple form and also (or rather as a consequence) that $\int_V |\varphi(\mathbf{x}, x_0)|^2 d\mathbf{x}$ may be directly interpreted as the probability of finding the particle within the volume V at time x_0 . In these various wave functions the scalar product takes the form:

$$(6) \quad (f_1, f_2) = i \int \left(f_1^* \frac{\partial f_2}{\partial x_0} - \frac{\partial f_1^*}{\partial x_0} f_2 \right) d^3\mathbf{x} \equiv i \int f_1^* \overleftrightarrow{\frac{\partial}{\partial x_0}} f_2 d\mathbf{x} = \int \varphi_1^* \varphi_2 d\mathbf{x} = \int \tilde{\varphi}^* \tilde{\varphi}_2 d\mathbf{p}.$$

We shall also need the asymptotic form of a solution of the Klein-Gordon equation for large times. This may be obtained by the method of stationary

phase:

$$\varphi(\mathbf{x}, t) = (2\pi)^{-3} \int \tilde{\varphi}(\mathbf{p}) \exp[i(\mathbf{p} \cdot \mathbf{x} - \sqrt{\mathbf{p}^2 + m^2} t)] d\mathbf{p}.$$

The phase in the integral is stationary for:

$$(7) \quad \frac{\mathbf{p}}{\sqrt{\mathbf{p}^2 + m^2}} t = \mathbf{x} \quad \text{or} \quad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2}} \quad \text{with} \quad \mathbf{v} = \frac{\mathbf{x}}{t}.$$

Therefore, if $\tilde{\varphi}(\mathbf{p})$ is sufficiently often continuously differentiable, the value of $\varphi(\mathbf{x}, t)$ is determined by the values of $\tilde{\varphi}(\mathbf{p})$ in the neighborhood of $\mathbf{p} = (m\mathbf{v})/\sqrt{1 - v^2}$. Expanding around this point one gets as the first term in the asymptotic expansion in powers of $1/t$.

$$(8) \quad \varphi(\mathbf{x}, t) \approx i^{\frac{3}{2}} \gamma (m\gamma/t)^{\frac{3}{2}} \tilde{\varphi}(m\gamma\mathbf{v}) \exp[-i\gamma^{-1}mt]; \quad \gamma = \sqrt{1 - v^2}.$$

In order to get information about the type of representation in the subspaces corresponding to more than 1 particle we need some results of the general theory of collision processes.

2'2. General collision theory. — The natural and general concepts of collision theory (whether we deal with many channel reactions in wave mechanics or with problems in quantum field theory) are the following:

a) The notion of localized states *i.e.* a criterion which tells whether a state Φ is at time t localized in a region V .

b) The notion of an asymptotic product between states: If Φ_1 and Φ_2 are any two states which are localized at time t in the regions V_1 and V_2 and if V_1 and V_2 are far apart, then there exists in nature also a state Φ in which we have the situation described by Φ_1 within V_1 as well as the situation Φ_2 within V_2 and nothing outside the regions V_1 and V_2 . We will call Φ the product state at time t between Φ_1 and Φ_2 :

$$\Phi = \Phi_1 \overset{(t)}{\wedge} \Phi_2.$$

Obviously this state Φ is not necessarily well defined operationally if V_1 and V_2 are close together but asymptotically, for large separation, it has an unambiguous physical meaning. We will find it convenient to define the *mathematical* operation $\overset{(t)}{\wedge}$ also between states which do not refer to well separated localization volumes but then we have to remember that it is only the asymptotic product which has a physical significance and that only this should enter into the final formulae.

2.3. *Connection between $\bigwedge^{(t)}$ and $\bigwedge^{(t+\tau)}$* : $\exp[iH\tau]\Phi$ is the state which at time $t' = t + \tau$ has the same spatial distribution as Φ at time t . If Φ is localized in V at time t , $\exp[iH\tau]\Phi$ will be localized in V at time $t + \tau$ and therefore

$$(9) \quad \left\{ \begin{array}{l} \exp[iH\tau](\Phi_1 \wedge^{(t)} \Phi_2) = \exp[iH\tau]\Phi_1 \wedge^{(t+\tau)} \exp[iH\tau]\Phi_2, \\ \text{or} \\ \Phi_1 \wedge^{(t+\tau)} \Phi_2 = \exp[iH\tau]\{\exp[-iH\tau]\Phi_1 \wedge^{(t)} \exp[-iH\tau]\Phi_2\}. \end{array} \right.$$

If the localization volumes of Φ_1 and Φ_2 are far apart then the interaction between the two subsystems is negligible, which means, that (for not too long time intervals τ) $\exp[iH\tau]\{\Phi_1 \wedge^{(t)} \Phi_2\}$ and $\exp[iH\tau]\Phi_1 \wedge^{(t)} \exp[iH\tau]\Phi_2$ is the same thing, or

$$\Phi_1 \wedge^{(t+\tau)} \Phi_2 \approx \Phi_1 \wedge^{(t)} \Phi_2.$$

From this we want to prove now that

$$(10) \quad \lim_{t \rightarrow \infty} \Phi_1 \wedge^{(t)} \Phi_2$$

exists (is a definite, finite vector in Hilbert space) for arbitrary normalizable component states Φ_1 and Φ_2 and that only the definition of the product between infinitely separated components enters into (10) in the limit. The limit is to be understood in the sense of strong convergence. The exact form of the assumption we make about the vanishing of the interaction for large distances in order to prove Eq. (10) is

$$(11) \quad \lim_{R \rightarrow \infty} R^n \|\exp[iH\tau](\Phi_1(R) \wedge \Phi_2) - (\exp[iH\tau]\Phi_1(R)) \wedge (\exp[iH\tau]\Phi_2)\| = 0$$

for some exponent $n > 1$ and for any fixed states Φ_1 , Φ_2 and any fixed value τ . In (11)

$$\Phi_1(R) = \exp[i\mathbf{P}R]\Phi_1.$$

Let us prove (10) from the assumption (11) for the case in which Φ_1 and Φ_2 are single particle states.

Let us do this for the case where Φ_1 and Φ_2 are single particle states. The spatial probability distribution of the two particles at large times is given by (8). If we write $\Phi_{1,2} = \sum \Phi_{1,2}^{(i)}$ where each $\Phi^{(i)}$ has a momentum space wave function $\varphi^{(i)}(\mathbf{p})$ which vanishes outside of a cell i , then for indices i and k which are not identical (or immediate neighbors) the localization volumes of $\Phi^{(i)}$ and $\Phi^{(k)}$ are far separated for large t .

We have

$$(12) \quad \Phi_1 \wedge^{(t)} \Phi_2 = \sum' \Phi_1^{(i)} \wedge^{(t)} \Phi_2^{(k)} + \sum \Phi_1^{(i)} \wedge^{(t)} \Phi_3^{(i)}.$$

The prime in the first sum means that $i = k$ (and more exactly also immediate neighbors) have been omitted. These terms are taken care of in the second sum.

Let v be the volume of the momentum space cells. The norm squared of $\Phi_1^{(i)} \wedge \Phi_2^{(i)}$ is proportional to v^2 . Furthermore $\Phi_1^{(i)} \wedge \Phi_2^{(i)}$ and $\Phi_1^{(k)} \wedge \Phi_2^{(k)}$ become orthogonal for large t if i and k are not identical (or immediate neighbors) because the localization volumes become disjoint. Therefore the norm squared of the second sum in (12) is essentially $\sum \Phi_1^{(i)} \cdot \Phi_2^{(i)} \cdot v^2$ which is proportional to v since the number of terms is proportional to $1/v$. By choosing a sufficiently fine cell division we can thus reduce the second term in (12) to an arbitrarily small contribution.

On the other hand, for any term in the first sum, the distance R_{ik} between the localization volumes grows proportional to t . Therefore, according to (10),

$$\lim_{t \rightarrow \infty} t^n \|\Phi^{(i)} \wedge^{(t+\tau)} \Phi^{(k)} - \Phi^{(i)} \wedge^{(t)} \Phi^{(k)}\| \rightarrow 0.$$

If $n \geq 1$ this implies

$$\|\Phi_1 \wedge^{(t_1)} \Phi_2 - \Phi_1 \wedge^{(t_2)} \Phi_2\| < \varepsilon \quad \text{if } t_1, t_2 > T(\varepsilon)$$

i.e. the existence of the strong limit of (11) for $t \rightarrow \infty$. The generalization of this to states Φ_1, Φ_2 which describe several particles is obvious.

We will from now on denote the operation $\lim_{t \rightarrow +\infty} \wedge^{(t)}$ by $\wedge^{(-)}$ and $\lim_{t \rightarrow -\infty} \wedge^{(t)}$ by $\wedge^{(+)}$.

2.4. Metric of asymptotic product states. — If the localization volumes of Φ_1 and Φ_2 at time t are far separated and the same is true about the localization volumes of Φ'_1 and Φ'_2 at that time, then from the physical meaning of the product combination it follows that

$$(12) \quad (\Phi_1 \wedge^{(t)} \Phi_2, \Phi'_1 \wedge^{(t)} \Phi'_2) \approx (\Phi_1, \Phi'_1)(\Phi_2, \Phi'_2) + (\Phi_1, \Phi'_2)(\Phi_2, \Phi'_1).$$

The reason is, that we only get a non-vanishing scalar product if either V_1 and V'_1 as well as V_2 and V'_2 overlap or if V'_1 and V'_2 as well as V_2 and V_1 overlap. At most one of these alternatives can hold if the separation is large.

Consequence:

$$(13) \quad (\Phi_1 \wedge^{(+)} \Phi_2, \Phi'_1 \wedge^{(+)} \Phi'_2) = (\Phi_1 \Phi'_1)(\Phi_2 \Phi'_2) + (\Phi_1 \Phi'_2)(\Phi_2 \Phi'_1)$$

for any states $\Phi_1 \dots \Phi'_2$.

2.5. Transformation properties under the invariance group. — « The theory is invariant under a certain transformation » means:

a) There is a unitary (or antiunitary) operator R . It gives to every state Φ the «transformed state» $R\Phi$ (active point of view in the terminology of M.-W.).

b) If Φ happens to be such that at some time t it is decomposed into two far separated subsystems (i.e. if $\Phi = \Phi_1 \overset{(t)}{\wedge} \Phi_2$ with localization volumes V_1, V_2 far apart) then

$$(14) \quad R\Phi \approx R\Phi_1 \overset{(t)}{\wedge} R\Phi_2.$$

This is a strong restriction since it shall hold for all t . Also it is clearly necessary. For, consider a state of a system which is at this moment in this room. If the prescription for transforming this state depended on the state in which a particle on the moon happens to be then clearly such an «invariance principle» would be useless.

The easiest way to satisfy b) is to require that R should commute with the Hamiltonian H . Then, if b) holds for one time it holds for all times. Most of the important invariance principles are of this type (translations in space and time, rotations in space, space inversion. On the other hand «pure Lorentz transformations» do not commute with H but they still satisfy the more general requirement b). Time reflection requires a separate discussion. There, instead of (14) we have

$$(15) \quad T(\Phi_1 \overset{(t)}{\wedge} \Phi_2) \rightarrow (T\Phi_1) \overset{(-t)}{\wedge} (T\Phi_2) \quad \text{for large separation.}$$

Consequences:

$$(16) \quad R(\Phi_1 \overset{(\pm)}{\times} \Phi_2) = R\Phi_1 \overset{(\pm)}{\times} R\Phi_2 \quad R \text{ any invariance transformation without time reflection, in particular } R = U(a, \wedge).$$

$$(17) \quad T(\Phi_1 \overset{(\pm)}{\times} \Phi_2) = (T\Phi_1) \overset{(\mp)}{\times} (T\Phi_2) \quad T \text{ time inversion.}$$

2'6. *Summary of results.* – From the single particle states we can form by the operation $\overset{(\times)}{\wedge}$ or by $\overset{(\times)}{\vee}$ the asymptotic many particle states. We make now the physical assumption that this procedure yields us a complete basis in \mathcal{H} i.e. that there are no states which are not interpretable in terms of asymptotic particle observations. Furthermore we add, for mathematical convenience, a single vacuum state Φ_0 which is supposed to be invariant under $U(a, \Lambda)$.

Then the structure of \mathcal{H} may be written as

$$(18) \quad \mathcal{H} = \Phi_0 \oplus \sum \mathcal{H}^{(j)} \oplus \sum \mathcal{H}_{(-)}^{(jk)} \oplus \dots,$$

where $\mathcal{H}^{(j)}$ are the subspaces corresponding to all different types (index j) of

single particles, $\mathcal{H}_{(-)}^{(jk)}$ is the subspace of $\Phi_1 \times \Phi_2$, $\Phi_1 \in \mathcal{H}^{(j)}$, $\Phi_2 \in \mathcal{H}^{(k)}$, i.e. the states corresponding to two incoming particles of types j and k respectively. The important point, following from the preceding discussion is now:

Theorem: $\mathcal{H}_{(-)}^{(jk)}$ is isomorphic to the direct product between the spaces $\mathcal{H}^{(j)}$ and $\mathcal{H}^{(k)}$ (or, if $j = k$, to the symmetrized direct product). The isomorphism refers to the algebraic operations (sum, scalar product) and to the transformation properties under the invariance group.

Thus, since $\mathcal{H}^{(j)}$ and $\mathcal{H}^{(k)}$ transform under the irreducible representations $D^{(j)}$, $D^{(k)}$ the space $\mathcal{H}^{(jk)}$ transforms under the direct product representation $D^{(j)} \times D^{(k)}$.

The extension to the higher subspaces $\mathcal{H}^{(jkl)}$, etc., is obvious. At first sight this theorem may be a little surprising because it means that the rep. $U(a, \wedge)$ in the whole space \mathcal{H} is exactly the same as that of a system of non-interacting fields $A_0^{(j)}(x)$ (one field for every particle type j).

In particular it means that if the Hamiltonian and the momentum operators are given in terms of some system of basic variables (e.g. in terms of a canonical system q_k , p_k of operators) the physical interpretation of which is unknown, then the single particle states are uniquely determined but the interaction between the particles is completely undetermined. Only when we know the connection between this basic set of operators and space-time measurements the interaction is fixed by giving $H(p_k, q_k)$.

2.7. « Incoming and outgoing fields ». — From the above theorem there follows an easy way to introduce a complete system of operators in \mathcal{H} . Take a system of free fields $A_j^{(\text{in})}(x)$, one field for each particle type. Each such field obeys the Klein-Gordon equation with the experimental mass of the particle in question and the free field commutation relations:

$$(19) \quad (\square - m_j^2)A_j^{(\text{in})}(x) = 0; \quad [A_j^{(\text{in})}(x)A_k^{(\text{in})}(x')] = i\delta_{kj}\Delta(x-x'; m_j).$$

The physical meaning of these « incoming fields » shall be the following (we drop the index j from now on). Let Φ be a single particle state corresponding to the wave function $f(x)$, Ψ an arbitrary state and

$$(20) \quad A_j^{(\text{in})}(t) = i \int \left(A^{\text{in}}(x) \frac{\partial f}{\partial x_0} - \frac{\partial A^{\text{in}}}{\partial x_0} f(x) \right) d^3x.$$

This quantity is actually independent of t as it is the scalar product of two solutions of the Klein-Gordon equation. Then

$$(21) \quad A_j^{(\text{in})}\Psi = \Phi \times \Psi,$$

i.e. $A_f^{(\text{in})}$ is a creation operator adding one more particle to the initial configuration.

Similarly one defines the « outgoing fields » $A^{(\text{out})}(x)$, the only difference being that we replace the minus sign by a plus sign in (21). Thus $A_f^{(\text{out})}$ adds a particle to the final configuration.

Notation. Let us use a complete orthonormal basis system of single particle states Φ_α (corresponding wave functions $f_\alpha(x)$). Instead of

$$\Phi_\alpha \times^{(-)} \Phi_\beta \dots \times^{(-)} \Phi_\delta,$$

we write $|\alpha\beta\dots\delta\rangle$, or sometimes simply $|\bar{a}\rangle$, abbreviating the set of indices $\alpha\beta\dots\delta$ which characterizes a configuration by a single letter. From (13) it follows that the states $|^{(-)}\bar{a}\rangle$ (and similarly the $|^{(+)}\bar{a}\rangle$) have simple orthogonality properties. We write then symbolically

$$(22) \quad \langle^{(-)}\bar{b} | ^{(-)}\bar{a}\rangle = \delta_{ab}; \quad \langle^{(+)}\bar{b} | ^{(+)}\bar{a}\rangle = \delta_{ab}.$$

S-matrix. Since the $|^{(-)}\bar{a}\rangle$ as well as the $|^{(+)}\bar{a}\rangle$ form a complete basis system in \mathcal{H} and since the scalar products between corresponding elements within these two systems are identical (eq. (22)), there is a unitary operator S mapping $|^{(+)}\bar{a}\rangle$ on $|^{(-)}\bar{a}\rangle$:

$$(23) \quad |^{(-)}\bar{a}\rangle = S |^{(+)}\bar{a}\rangle.$$

Alternatively, S may be defined by

$$(23') \quad A^{(\text{in})}(x) = S A^{(\text{out})}(x) S^{-1}.$$

The probability amplitude for observing a configuration b after the collision when one starts from an initial configuration a (S -matrix element) is

$$(24) \quad \langle^{(+)}\bar{b} | ^{(-)}\bar{a}\rangle = \langle^{(+)}\bar{b} | S | ^{(+)}\bar{a}\rangle = \langle^{(-)}\bar{b} | S | ^{(-)}\bar{a}\rangle.$$

2.8. *Invariance properties of S .* — According to (16) we have for every element R of the invariance group (not involving time reflection)

$$(25) \quad [SR] = 0$$

which gives of course the conservation laws for energy, momentum, angular momentum, parity.

The invariance under time reflection gives the principle of detailed ba-

lancing, which we formulate most simply for the S -matrix elements directly: Let a_{τ} be the configuration obtained from a by applying the time reflection to each single particle state contained in a separately (see M.-W.). Then (17) tells

$$(26) \quad T|\bar{a}\rangle = |a_{\tau}^{(+)}\rangle; \quad T|a^{(+)}\rangle = |\bar{a}_{\tau}\rangle.$$

Therefore, since T is antiunitary

$$(27) \quad S_{ba} \equiv \langle b^{(+)} | a^{(+)} \rangle = \langle (Tb^{(+)}) | (Ta^{(+)}) \rangle^* = \langle b_{\tau}^{(-)} | a_{\tau}^{(+)} \rangle^* = \langle a_{\tau}^{(+)} | b_{\tau}^{(-)} \rangle = S_{a_{\tau}b_{\tau}}.$$

2'9. *Connection to the Møller formal theory of scattering.* — It may be that there exists an hermitian operator H_0 such that (with a suitable definition of the product \wedge for small distances)

$$(28) \quad \exp[iH\tau]\Phi_1 \wedge^{(0)} \exp[iH\tau]\Phi_2 = \exp[iH_0\tau]\Phi_1 \wedge^{(0)} \Phi_2$$

or every pair of states Φ_1, Φ_2 . Then according to (9)

$$\Phi \wedge^{(t)} \Phi_2 = \exp[iHt] \exp[-iH_0t] \Phi_1 \wedge^{(0)} \Phi_2$$

and the existence of the limits $t \rightarrow \pm \infty$ implies the existence of the Møller « wave matrices »

$$\lim_{\pm \infty} \exp[iHt] \exp[-iH_0t].$$

This formalism is however not easily extended to the case of many channel reactions because there one would have to introduce a set of different operators H_0 which has several disadvantages. (See, e.g., H. EKSTEIN: *Phys. Rev.* **101**, 880 (1956)).

3. — The field operators and their assumed properties.

Let us consider again the simplest case of a classical relativistic field theory with interaction which is known to be meaningful namely the one with a single scalar field satisfying the equation

$$(29) \quad (\square - m^2)A(x) = gA^3(g).$$

Challenge: To construct a quantum theory which corresponds to this classical model. If this can be done at all in any sensible way then the result

should incorporate the concepts discussed in the last section. Beyond this we shall want that the basic quantities of the theory, in terms of which everything else has to be expressed ultimately, is a set $A(x)$ (one quantity A to every point x in space time). Now the first (and fundamental) difficulty is, that a single $A(x)$ cannot be an honest observable both for physical and mathematical reasons. It should correspond to something like the measurement of the force on a test body in a single space-time point and this is impossible by the uncertainty principle (see N. BOHR and L. ROSENFELD: *Dan. Mat. Fys. Medd.*). The best one can hope for, is that the weighted averages of $A(x)$ over finite space regions are observable, *i.e.* quantities of the type $\int g(x)A(x)dx$, where $g(x)$ is some weight function.

Question: Is it necessary to average over a 4-dimensional region or is it legitimate to talk about measurements at an exact time t ?

The answer, based on the experience which KÄLLÉN, THIRRING and others have gained in their calculations seems to be: in a model like the one above a 4-dimensional averaging is necessary as soon as $g \neq 0$ in contrast to the free field case, in which we know that a 3-dimensional smearing is sufficient. To be on the safe side therefore we start with the most restrictive class of test functions g , the class \mathcal{D} of infinitely differentiable functions vanishing outside of a finite region in space time and require only:

- 1) To every real function $g(x) \in \mathcal{D}$ there is an hermitian operator

$$\int A(x)g(x)d^4x \quad \text{in} \quad \mathcal{H}.$$

In the terminology of the Gårding-Lyons course one would say that $A(x)$ is an operator valued distribution. We will henceforth take care of this mostly by a mental note and talk for the sake of simplicity about the « operator » $A(x)$. But, of course, the distribution character of $A(x)$ means, that the cube $A^3(x)$ appearing in the classical field equation is undefined. In fact the central question in the whole game is: can one define something corresponding to $A^3(x)$ by some limiting procedure? But this question can certainly not be answered at the beginning and therefore we cannot formulate the equation of motion.

A second requirement for the $A(x)$ is obvious. They shall furnish a complete description:

- 2) The system $A(x)$ is a complete system of operators in \mathcal{H} ; or, mathematically: the space \mathcal{H} is irreducible with respect to the ring R generated by the $A(x)$.

In particular, therefore, the Lorentz operators $U(a, \wedge)$ must be expressible as functions of the $A(x)$.

As a third property we want that the set $A(x)$ transforms under the Lorentz group like a scalar field, *i.e.*

$$(3) \quad U(a, \Lambda)A(x)U^{-1}(a, \Lambda) = A(\Lambda x + a).$$

The other two general properties which we want the field to have are perhaps more problematical. They are intended to express the causal structure of the theory and they are about as extreme as it is possible to be without stumbling into one of the known pitfalls. May be they are too strong. But it is in the spirit of this conservative approach not to give up anything unless one is forced to do so and quite apart from this we do not know at the moment of any natural way to loosen these requirements. Thus we require:

4) There should be an equation of motion, such that $A(x)$ is determined by the set $A(y)$ with $y \in \mathcal{B}$, where \mathcal{B} is any space time region which intercepts all the backward (or forward) time-like rays from the point x .

Mathematically: Call a region \mathcal{B}' dependent on \mathcal{B} if all the backward (or all the forward) time-like rays starting from points in \mathcal{B}' pass through \mathcal{B} . Call $R_{\mathcal{B}}$ the ring generated by the $A(x)$ with $x \in \mathcal{B}$. Then we require that $R_{\mathcal{B}'}$ is a subring of $R_{\mathcal{B}}$ whenever \mathcal{B}' is dependent on \mathcal{B} . Clearly the hyperbolic differential equation (29) would satisfy this requirement. We cannot use a differential equation unfortunately, because the field needs a 4-dimensional smearing and is undefined at an exact time.

5) $[A(x), A(y)] = 0$ for space-like separation $x - y$.

The simple physical idea of that is that if f_1 and f_2 are two test functions with supports in \mathcal{B}_1 and \mathcal{B}_2 and if it is true that $\int A(x)f_1(x)dx$ corresponds to a measurement which affects exactly the region \mathcal{B}_1 , then this measurement should not interfere with $\int A(x)f_2(x)dx$ as long as the regions \mathcal{B}_1 and \mathcal{B}_2 are totally space-like with respect to each other.

4. - Physical interpretation.

In the classical model described by (29) the density of the interaction energy is $g \cdot A^4(x)$, *i.e.* positive for positive g . Therefore if there exists a corresponding quantum theory we will not expect any binding effects. In other words we expect that there will be only one type of particle described by it.

The situation is then particularly simple in so far as we have to deal only with *one* field $A^{\text{in}}(x)$, one $A^{\text{out}}(x)$ and one $A(x)$. What is the connection between these fields?

4.1. *Formal approach (asymptotic condition).* — Let us start with some formal manipulations. They disregard the warning signs put up in Section 3 but they have perhaps some heuristic value for suggesting the connection between A^{in} and A .

Suppose we quantize (29) according to the canonical scheme (see KÄLLÉN's lecture) and introduce the «free Hamiltonian»

$$H_0 = \frac{1}{2} \int (\text{grad } A)^2 + \pi^2 + m^2 A^2 d^3x,$$

where A and π are the canonical variables at time zero and m is the «experimental» mass of the particle. If we followed uncritically the procedure of the formal scattering theory of wave mechanics we would assume that

$$(30) \quad R = \lim_{t \rightarrow -\infty} \exp[iHt] \exp[-iH_0 t]$$

exists and is a unitary operator (the Møller wave matrix). Then, transforming $A(x)$ and $\pi(x)$ by R we would obtain a second canonical system of operators

$$(31) \quad A^{\text{in}}(x) = RA(x)R^{-1}; \quad \pi^{\text{in}}(x) = R\pi(x)R^{-1}.$$

Since according to (30)

$$(32) \quad \exp[iH\tau]R = R \exp[iH_0\tau],$$

the equation of motion of the $A^{\text{in}}(x)$ would be the *free* Klein-Gordon equation if we define A^{in} for arbitrary times by

$$(33) \quad A^{\text{in}}(x, t) = \exp[iHt]A^{\text{in}}(x) \exp[-iHt]$$

(H is the *total* Hamiltonian).

This justifies the name A^{in} for the quantities defined by (31). We may now use (32) to express the connection between A^{in} and A at an arbitrary time x_0 :

$$A^{\text{in}}(x) = R \exp[iH_0 x_0] \exp[-iH x_0] A(x) \exp[iH x_0] \exp[-iH_0 x_0] R^{-1}.$$

This gives in the limit $x_0 \rightarrow -\infty$

$$(34) \quad \lim_{x_0 \rightarrow -\infty} (A(x) - A^{\text{in}}(x)) \rightarrow 0.$$

(34) is, of course, a mere symbolism. What shall be the exact meaning of

it? First we must imagine (34) multiplied with some test function and integrated. This is harmless. Secondly we must say what the limit means. In wave mechanics the corresponding limit relations are true in the sense of strong convergence. This would mean: apply the (smeared) operator on the left hand side of (34) to any fixed state vector Ψ . Then the sequence of image vectors shall go towards zero *in length* as $x_0 \rightarrow -\infty$. Now it is easy to see that strong convergence in (34) is impossible for a local field theory, the reasons are:

1) The component of $\int f(\mathbf{x}, x_0 - t) A(\mathbf{x}) d^4x |0\rangle$ which is orthogonal to the one particle states does not decrease in norm as $t \rightarrow -\infty$. Therefore the strong convergence in (34) can only be true if $A(\mathbf{x})|0\rangle$ is a one particle state.

2) If the $A(\mathbf{x})$ satisfies local commutation relations, $A(\mathbf{x})|0\rangle$ cannot be a one particle state unless the interaction vanishes.

Although both these statements were given in my Copenhagen lecture notes of 1953-54 I had forgotten the first one of them when I wrote the paper which appeared in *Dan. Mat. Fys. Medd.* (1955). As a consequence there is a wrong formulation of the asymptotic condition in that paper.

The first correct formulation is due to LEHMANN, SYMANZIK and ZIMMERMANN (*Nuovo Cimento* (1955)). It is (see also LEHMANN's lectures)

$$(35) \quad \lim_{t \rightarrow -\infty} \langle \Phi | A_f(t) | \Psi \rangle \rightarrow \langle \Phi | A_f^{\text{in}} | \Psi \rangle,$$

with

$$(36) \quad A_f(t) = i \int_{x_0=t} (A(x) \dot{f}(x) - \dot{A}(x) f(x)) d^3x,$$

where f is a positive energy solution of the Klein-Gordon equation and Φ, Ψ two arbitrary states. The essential difference between (35) and the customary relations in wave mechanics is that only the *matrix elements* of $A_f(t)$ between two fixed states are assumed to converge whereas the *vectors* $A_f(t)|\Psi\rangle$ need not approach any limit as $t \rightarrow -\infty$. Mathematically: Only weak convergence is required. The justification of (35) by L.S.Z. was that its application gave the same results for the S -matrix in perturbation theory as Dyson's technique.

To summarize. Add to the 5 requirements for $A(x)$ which were stated in Section 3 the « asymptotic condition » (35) namely *a*) that the weak limits of $A_f(t)$ exist for $t \rightarrow \pm \infty$ and *b*) that the incoming (or outgoing) fields defined by these limits satisfy the proper commutation relations (19). This is one way of making sure that the theory has an interpretation in terms of particles.

Problem. Find an operator field $A(x)$ which satisfies all these conditions (except the trivial solution $A = A^{\text{in}} = A^{\text{out}}$). Essentially nothing has been done about this «existence problem» so far. A more modest problem but still tough enough: Find the consequences which these requirements for $A(x)$ imply for the S -matrix and related quantities.

Such questions will be discussed in the talks by LEHMANN and in WIGHTMAN's lectures on analytic properties of vacuum expectation values.

4.2. «Localized states». — In the previous Section we have regarded the asymptotic conditions as a *postulate* which is imposed on the field in addition to the general requirements listed in Section 3. The analogous procedure in a wave mechanical scattering problem would be to *postulate* that the limits of $\exp[iHt] \exp[-iH_0 t]$ exist for $t \rightarrow \pm \infty$ and define unitary operators.

Now, of course, these asymptotic properties for $|t| \rightarrow \infty$ are a consequence of the vanishing of the effective interaction between two subsystems as their separation $R \rightarrow \infty$. The way how to translate the asymptotic properties for large R at finite times into asymptotic properties for large $|t|$ has been described in very general form in Section 2. If we apply this method to the case of field theory we can learn:

a) To replace the asymptotic condition by a requirement for products of field operators at a finite time.

b) To treat collision problems also in such theories in which there is no direct co-ordination between the set of basic fields and the set of particle types which are described by the theory (e.g. a theory with one basic field $A(x)$ which describes 25 different particles and therefore has 25 different incoming fields $A^{\text{in}}(x)$).

c) To construct the incoming states by other limiting procedures than (35) which may occasionally be useful.

If \mathfrak{B} is a space time region we introduce the symbol $\mathcal{R}_{\mathfrak{B}}$ to denote the set of operators which are functions of the $A(x)$ with x restricted to \mathfrak{B} . By $\mathcal{H}_{\mathfrak{B}}$ we denote the subspace of states obtained by the application of these operators on the vacuum state. We introduce a (partial) physical interpretation of the field quantities $A(x)$ by assuming that the states of $\mathcal{H}_{\mathfrak{B}}$ can not be distinguished from the vacuum by means of observations in a region which is far away from \mathfrak{B} in a space like direction. In other words $\mathcal{H}_{\mathfrak{B}}$ shall contain the states which are experimentally localized in \mathfrak{B} .

This interpretation is only possible if

$$(37) \quad \langle 0 | q_R \cdot q' | 0 \rangle \approx \langle 0 | q_R | 0 \rangle \langle 0 | q' | 0 \rangle \quad \text{for large } R$$

with

$$(38) \quad g_R = \exp[i\mathbf{PR}]g \exp[-i\mathbf{PR}]; \quad g, g' \in \mathcal{R}_3.$$

In fact we shall assume that the difference between the two sides of equation (37) decreases faster than any power of R .

Remark: 1) In the theory of relativistic free fields this difference decreases as $\exp[-mR]$ (m being the smallest mass).

2) In the case of a field theory with interaction satisfying the requirements of Section 3 it can be *proved* that the difference decreases faster than any power, provided the energy-momentum spectrum is of the type discussed in the Section 2 and provided that no particles with zero mass occur in the theory. If there are particles of zero mass then (40) is not true and the following approach must be modified.

We reformulate condition (37) (or rather a generalization of it) in terms of vacuum expectation values of products of field operators.

For convenience we use regularized fields $C_f(x) = \int f(x-y)A(y)dy$, $f \in \mathcal{D}$ rather than the $A(x)$ themselves. These objects have a meaning at a point. We want to describe the behaviour of the quantity $\langle 0|C_1(x_1) \dots C_n(x_n)|0 \rangle$ if some of the points x_j have large space like distances from the other points. Let us take all the times equal ($x_{10} = x_{20} = \dots = x_{n0}$) and consider for example a division of the n points into the three groups (x_1, x_3) (x_2) $(x_4 \dots x_n)$. Imagine the points within each group being kept at a fixed constant distance from each other while the three groups are moved towards infinity in different directions of space. Then, according to the physical picture underlying (40) we want that $\langle 0|C_1(x_1) \dots C_n(x_n)|0 \rangle$ approaches the product $\langle 0|C_1(x_1)C_3(x_3)|0 \rangle \cdot \langle 0|C_2(x_2)|0 \rangle \langle 0|C_4(x_4) \dots C_n(x_n)|0 \rangle$ in such a way that the difference between the two expressions vanishes faster than any power with increasing distances between the groups. For a general and compact formulation it is convenient to introduce the «truncated part» $(C(x_1) \dots C(x_n))_T$ of a vacuum expectation value by subtracting from the latter all those terms which are independent of the separation in some particular grouping. A recursive definition is

$$(39) \quad (C(x))_T = \langle 0|C(x)|0 \rangle,$$

$$(40) \quad \langle 0|C_1(x_1) \dots C_n(x_n)|0 \rangle = \sum (C_j(x_j) \dots)_T (C_k(x_k) \dots)_T \dots,$$

where the sum runs over all groupings of the n points.

Example. For $n=2$ (39) and (40) give

$$C(x_1)C(x_2)_T = \langle 0|C(x_1)C(x_2)|0 \rangle - \langle 0|C(x_1)|0 \rangle \langle 0|C(x_2)|0 \rangle.$$

Inserting this in the formula (40) for $n=3$ we obtain the definition of $(C(x_1)C(x_2)C(x_3))_T$. The postulate, by which we want to replace and generalize the asymptotic condition (35) is now

$$(41) \quad \lim_{R \rightarrow \infty} R^m (C_1(x_1) \dots C_n(x_n))_T \rightarrow 0$$

for any power m , if the times $x_{10} \dots x_{n0}$ are equal and if R is the radius of the smallest sphere enclosing the n points x_i . In other words, the truncated parts shall decrease faster than any power when any separation between its points gets large. One may call (41) the « *spatial asymptotic condition* » and (35) the « *as. cond. in time* ». From the point of view of axiomatics (41) is advantageous because unlike (35) it does not need an advance knowledge of the particle types which are described by the theory.

Remark. While (38) can be proved from the general properties of $A(x)$ and the spectral conditions this is not true so far of (41). The mentioned proof does not work for a division of the x_j into more than two groups which are moved simultaneously to infinity.

4.3. *Construction of states with given initial or final configurations.* — We want to show that if (41) is satisfied we can construct to every particle type (discrete eigenvalue of P_μ^2) an incoming field. In fact two methods for this construction will be given. One is a generalization of (35) and uses weak convergence, the other one, first suggested in field theory by H. EKSTEIN, uses strong convergence.

Let us first introduce a terminology. If we have a set of operators $B(x)$ in \mathcal{H} (one operator for every space-time point) we will call this a *field*, if it transforms correctly under translations:

$$\exp[-iPa]B(x)\exp[iPa] = B(x+a).$$

Nothing is assumed about the transformation properties under the homogeneous Lorentz group. It is not essential for our purposes to distinguish between the case where $B(x)$ is a honest operator and the one where it is a distribution, because in the latter case we can always regularize it (as done above in passing from $A(x)$ to $C(x)$). Consider fields of the type

$$(42) \quad B(x) = \sum \int g^{(n)}((x-y_1), \dots, (x-y_n)) A(y_1) \dots A(y_n) dy_1 \dots dy_n$$

(polynomials in the basic field) with coefficient functions $g^{(n)}(z_1 \dots z_n)$ which have a compact support in the time co-ordinates and which vanish faster than

any power of $|z_j|$ when any one of the $|z_i|$ gets large (*). It is easy to see that equations (39), (40), (41) still hold if the $C_i(x)$ there are replaced by any such $B_i(x)$. We will call therefore such fields «almost local». The only thing which is important about this class of almost local fields is that it contains the (regularized) basic field and that (41) holds when the C_i are replaced by any B_i from this class. We might then also take these properties directly as the definition of the class of almost local fields.

The main tool for the discussion of asymptotic properties for large times is the following:

THEOREM. () Let $B_i(x)$ be a set of almost local fields with $\langle 0 | B_i(x) | 0 \rangle = 0$, $f_i(x)$ a set of normalizable, positive energy solutions of Klein-Gordon equations to arbitrary mass values m_i . Put**

$$b_i(t) = \int_{x_0=t} B_i(x) f_i(x) d^3x.$$

a) Consider the vectors

$$\Phi(t) = b_1(t) b_2(t) \dots b_n(t) |0\rangle; \quad \Phi'(t) = b_r(t) b_s(t) \dots b_z(t) |0\rangle,$$

where r, s, z are some permutations of $1, 2, \dots, n$. Then the norm of the difference of Φ and Φ' vanishes as $t^{-\frac{n}{2}}$ in the limit $t \rightarrow \infty$:

$$\lim_{|t| \rightarrow \infty} \|\Phi(t) - \Phi'(t)\| \rightarrow ct^{-\frac{n}{2}}.$$

$$b) \quad \lim_{t \rightarrow \pm \infty} \langle 0 | b_1^\dagger(t) \dots b_n^\dagger(t) \dots b_{n+1}(t) \dots b_{2n}(t) | 0 \rangle \rightarrow \lim \sum \prod \langle 0 | b_i^\dagger(t) b_k(t) | 0 \rangle.$$

The sum runs over all possible pairings of one factor b^\dagger with one factor b .

THEOREM 2. a) If $Q^{(j)}(x)$ are almost local fields which, if applied on the vacuum, give one-particle-states (particle type j), then the strong limits

$$(43) \quad \lim_{t \rightarrow \pm \infty} Q_{f_{j_1}}^{(i_1)}(t) \dots Q_{f_{j_n}}^{(j_n)}(t) | 0 \rangle$$

exist and define the incoming (resp. outgoing) n -particle states.

b) The metric properties and the transformation properties of these states under the inhomogeneous Lorentz group are as discussed in Section 2. Here f_j is a Klein-Gordon wave function to mass m_j and $Q_j = i \int (\dot{Q} \dot{f} - Q f) d^3x$.

(*) To be completely safe we should use the regularized fields C instead of A on the right hand side of (42).

(**) For a proof see R. HAAG: *Phys. Rev.*, **112**, 669 (1958).

Remark. One may ask whether almost local operators exist for which $Q(x)|0\rangle$ is a one particle state. Since we know that there exist one particle states which are localized in the sense of NEWTON and WIGNER the definition of localized states in field theory adopted in this Section would be inconsistent if no almost local creation operators existed. Apart from this we have a definite hope to be able to prove the existence of the $Q^{(j)}(x)$ from other principles.

Instead of building up an incoming n -particle state by applying n factors $Q_f(t)$ on the vacuum as done in (43) we could probably just as well obtain it by applying a single factor $Q_f(t)$ on an incoming $(n-1)$ -particle state. In going over from Theorem 2 to this successive construction we could only run into trouble if the difference vector

$$Q_{f_1}(t) \dots Q_{f_n}(t)|0\rangle - \Phi_{(n-1)}^{\text{in}}$$

(which by the Theorem 2 decreases in length towards zero as $t \rightarrow -\infty$) would move out of the domain of $Q_{f_1}(t)$. This appears completely unreasonable to a physicist. We will disregard such fine points here but as a reminder that a more careful investigation is needed if one really wants to establish the two following results we will not call them theorems but «statements».

Statement I. If $Q^{(j)}(x)$ is an almost local field of creation operators for particle j (as in Theorem 2) and

$$(44) \quad \langle p|Q(0)|0\rangle = (2\pi)^{-\frac{1}{2}}F(p),$$

then $Q_f^{(j)}(t)$ converges strongly towards $A_g^{\text{in}(j)}$ where

$$(45) \quad \tilde{g}(p) = F(p)\tilde{f}(p).$$

This is essentially Theorem 2 apart from the mentioned domain problem. More remarkable is the next statement which establishes the connection with (35) and its generalizations for the case of composite particles (*).

Statement II. If $B(x)$ is any almost local field with

$$(46) \quad \langle 0|B(x)|0\rangle = 0; \quad \langle p|B(0)|0\rangle = (2\pi)^{-\frac{1}{2}}F(p) \neq 0,$$

where $|p\rangle$ is the state of one particle of type j and momentum p , then for

(*) These have been given by W. ZIMMERMANN (preprint, 1958) by a different method.

any two fixed states Ψ_1, Ψ_2

$$(47) \quad \lim_{t \rightarrow -\infty} \langle \Psi_1 | B_f(t) | \Psi_2 \rangle = \langle \Psi_1 | A_g^{\text{in}(f)} | \Psi_2 \rangle,$$

if f is a wave function to mass m_f . g is again given by (45).

Proof (up to domain questions). Suppose Ψ_2 is an incoming n -particle state. Replace it by expression (43) at time t . Use Theorem 1-a) to shift $B_f(t)$ to the extreme right, and statement I to get rid of the $Q(t)$ again. The problem is thus reduced to the discussion of the limit of $\langle \Phi | B_f(t) | 0 \rangle$ with Φ fixed. Suppose Φ is an incoming m -particle state. Replace it again by (43) at time t . Then we have $\langle 0 | Q_{f_1}^+ \dots Q_{f_m}^+ B_f(t) | 0 \rangle$. According to Theorem 1-b) this vanishes in the limit $t \rightarrow -\infty$ unless $m=1$. So only the 1-particle contribution of $B_f(t) | 0 \rangle$ survives and this is, by (46), (45) equal to $A_g^{\text{in}} | 0 \rangle$.

The statement II is somewhat surprising because it is so extremely easy to find almost local fields $B(x)$ which satisfy (46). For instance the basic field $A(x)$ itself is a candidate unless its matrix element between vacuum and the i -particle state in question vanishes. If we put $B(x) = A(x)$ we also have $F(p) = \text{const}$ because $A(0)$ is invariant under homogeneous Lorentz transformations. In a theory with only one type of particle the customary normalization of A is such that $F(p) = 1$. Then (47) reduces to (35).

We also see from (47) that it is possible to obtain from one $B(x)$ several incoming fields $A^{\text{in}(f)}$ as limits; the resulting limit depends on the mass value to which the wave function f belongs.

Summary. - Altogether we have formulated the following requirements for the quantities with which we deal in field theory:

- 1) Spectral conditions for $U(a, \wedge)$ (see Section 2).

A weaker form of these conditions is the following.

- 1a) The spectrum of the energy-momentum vector p should lie entirely inside of a hyperboloid $p^2 + m_0^2 = 0$; $p_0 > 0$. (Apart from the vacuum state $p = 0$).

- 2) The conditions for the basic field $A(x)$ (Section 3).

- 3) The asymptotic condition in space *i.e.* equation (41). These conditions are not completely independent and therefore not yet optimal for an analysis of their consequences. We can, *e.g.* obtain at least a part of 3) from 1a) and 2) (see Section 4) and conversely, we obtain from 2) and 3) above at least the following property of the spectrum: If p_1 and p_2 are in the spectrum then $p_1 + p_2$ is also in the spectrum.

On the other hand WIGHTMAN has given some simple models of field theories

which satisfy 1a), 3) and all requirements contained in 2) except, possibly, the existence of an equation of motion. They are obtained by putting

$$(48) \quad A(x) = : A_0(x)^n :$$

where A_0 is a free field and the dots mean the Wick product. (Split A_0 in creation and destruction part and rearrange the n -th power so that the destruction operators stand on the right in every term). The model (48) does not have any 1-particle states and is therefore completely unphysical. This may serve as a warning that possibly the requirements 1a), 2), 3) are not sufficient to guarantee a physically acceptable theory.

Scattering Matrix and Field Operators.

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In these lectures we shall be concerned with the following problem: Given the masses, spins, etc., of relativistic particles which interact, what properties of the scattering matrix which describes the interaction can we deduce from the general principles of the quantum theory of fields? These general principles have been formulated in an axiomatic manner by a number of people; most concisely by WIGHTMAN [1].

Without going into details, let me remind you of their main contents: The interaction is formulated in terms of field operators which I denote by $A(x)$ or $\psi(x)$.

These operators are assumed to possess:

- 1) An asymptotic behavior in the time component x_0 of their argument vector which insures the possibility of a particle interpretation.
- 2) Correct transformation properties under the inhomogeneous Lorentz group.
- 3) Vanishing commutators (or anticommutators) for space-like separation of their arguments. We believe that this property corresponds to causality.

Why do we pursue the question which I formulated?

It is hoped that by obtaining explicit properties of the scattering matrix we may:

- a) Correlate and thereby partially understand experimental results.
- b) Test the correctness of the axioms by confronting them with empirical facts.

Another motivation is of course the lack of anything better. That is, despite many attempts, nobody has succeeded in formulating *specific* interactions (where *e.g.* the energy-momentum vector P_μ is given in terms of interacting field operators) in such a way that reliable quantitative calculations can be made, in the case of strongly interacting relativistic particles.

The present situation with respect to our problem is such that a number of results have been obtained during the past few years. Most of these results are known as *dispersion relations* whose study in the present context was initiated by GOLDBERGER [2]. However, we are very far from a complete or systematic answer to our question.

Due to this state of affairs (and also due to my inabilities) I shall not attempt in these lectures a rigorous mathematical analysis. Rather I shall outline the ideas and methods and I hope to do it in such a way that the steps which must be taken to justify our procedure are more or less apparent. I shall divide the lectures into three parts. We need:

- I) The connection between scattering matrices and field operators.
- II) A mathematical representation of causal commutators. With these tools we may then discuss the results which are known so far, *i.e.*
- III) Derivation of dispersion relations.

1. - The connection between scattering matrices and field operators.

The initial and final particles in a scattering or production process are described by means of *free* (*i.e.* non-interacting) fields. That is, in the case of neutral scalar particles with experimental mass m by fields $A_{\text{in/out}}(x)$ which satisfy

$$(1) \quad (\square - m^2)A_{\text{in}} = 0; \quad [A_{\text{in}}(x), A_{\text{in}}(y)] = i\Delta(x-y),$$

(the same relations for the outgoing field).

The scattering matrix is then given by

$$(2) \quad S_{(\alpha)(\beta)} = \langle (\alpha)_{\text{out}} | (\beta)_{\text{in}} \rangle = \langle (\alpha)_{\text{in}} | S | (\beta)_{\text{in}} \rangle,$$

where the unitary operator S transforms the incoming into outgoing fields according to

$$(3) \quad A_{\text{out}}(x) = S^+ A_{\text{in}}(x) S.$$

For later use, I collect first some elementary formulae of the theory of free fields.

In eq. (2) the indices (α) , (β) refer to states of a complete orthonormal system of state vectors which spans the Hilbert space of our system.

In practice, plane waves-corresponding to particles with definite momenta are used to characterize the states. An incoming state of n particles with momenta $k_1 = (\mathbf{k}_1, \sqrt{\mathbf{k}_1^2 + m^2})$, k_2, \dots, k_n is obtained from the vacuum state $|0\rangle$ by applying creation operators $a_{\text{in}}^+(k)$

$$(4) \quad |k_1 \dots k_{n_{\text{in}}}\rangle = a_{\text{in}}^+(k_1) \dots a_{\text{in}}^+(k_n) |0\rangle.$$

The operators $a_{\text{in}}^+(k)$ and their adjoints $a_{\text{in}}(k)$ are defined by the Fourier decomposition of the field $A_{\text{in}}(x)$,

$$(5) \quad A_{\text{in}}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^4k \exp[-ikx] \delta(k^2 - m^2) A_{\text{in}}(k) =$$

$$(5') \quad = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3k}{2k_0} \{ \exp[-ikx] a_{\text{in}}(k) + \exp[ikx] a_{\text{in}}^+(k) \},$$

with

$$(6) \quad a_{\text{in}}(k) = A_{\text{in}}(k), \quad a_{\text{in}}^+(k) = A_{\text{in}}(-k) \quad \text{for } k_0 > 0.$$

We have the invariant relations

$$(7) \quad [a_{\text{in}}(k), a_{\text{in}}^*(k')] = 2k_0 \delta(\mathbf{k} - \mathbf{k}'); \quad [a_{\text{in}}(k), a_{\text{in}}(k')] = 0.$$

Corresponding relations hold of course for the outgoing field.

For a more satisfactory mathematical treatment one may replace the continuous system of plane waves and corresponding non-normalizable state-vectors by a discrete system of wave functions $f_{\lambda}(x)$ which are positive frequency solutions to the Klein-Gordon equations.

It is essential that in the two orthonormal systems $\Phi_{\text{in}}^{(\alpha)}$ and $\Phi_{\text{out}}^{(\lambda)}$, the vacuum and one-particle states are identical (with an appropriate choice of phase factors),

$$(8) \quad |0_{\text{in}}\rangle = |0_{\text{out}}\rangle = |0\rangle, \quad |k_{\text{in}}\rangle = |k_{\text{out}}\rangle = |k\rangle.$$

We note now that the expansion (5') is not restricted to the case of free fields. Using a discrete system of wave functions $f_{\lambda}(x)$ we can also write

$$(9) \quad A(x) = \sum_{\lambda} \{ f_{\lambda}(x) A^{\lambda}(x_0) + f_{\lambda}^*(x) A^{\lambda+}(x_0) \},$$

with the inversion

$$(10) \quad A^\alpha(x) = i \int d^3x A(x) \frac{\overleftrightarrow{\partial}}{\partial x_0} f_\alpha^*(x) = i \int d^3x \left\{ A(x) \frac{\partial}{\partial x_0} f_\alpha^*(x) - \frac{\partial}{\partial x_0} A(x) f_\alpha^*(x) \right\}.$$

In the case of free fields, A_{in}^α or A_{out}^α are of course time-independent.

To connect the scattering operator S with the interacting field $A(x)$, the latter must be related to the incoming and outgoing fields. It is customary to look upon $A(x)$ as an operator which interpolates for finite times between the operators $A_{\text{in}}^\alpha(x)$ which describe a physical situation in the limits $x_0 \rightarrow \pm \infty$.

We are led in this manner to the *asymptotic condition* for the field operator $A(x)$

$$(11) \quad \lim_{x_0 \rightarrow \pm \infty} \langle \Phi | A^\alpha(x_0) | \Psi \rangle = \langle \Phi | A_{\text{in}}^\alpha | \Psi \rangle$$

for arbitrary states Φ, Ψ .

A more detailed analysis of these relations is contained in HALG's lectures.

As a consequence of (11) the following formulae hold for the matrix elements of $A(x)$ between the vacuum and *one-particle states*:

$$(12) \quad \langle 0 | A(x) | \alpha \rangle = \langle 0 | A_{\text{in}}(x) | \alpha \rangle = f_\alpha(x).$$

Let me now illustrate by a typical example how the scattering matrix can be expressed—making use of the asymptotic condition (11)—in terms of Heisenberg operators.

Consider the scattering of two particles

$$p + k \rightarrow p' + k'$$

with the indicated initial and final momenta. Let k, k' refer to a neutral spin 0 particle with mass m .

The corresponding scattering matrix element is

$$(13) \quad \langle p' k'_{\text{out}} | p k_{\text{in}} \rangle = \langle p' | a_{\text{out}}(k') | p, k_{\text{in}} \rangle = \lim_{t \rightarrow +\infty} \langle p' | A(\mathbf{k}', t) | p k_{\text{in}} \rangle = \\ = \int_{-\infty}^{+\infty} dt \frac{\partial}{\partial t} \langle p' | A(\mathbf{k}', t) | p k_{\text{in}} \rangle + \lim_{t \rightarrow -\infty} \langle p' | A(\mathbf{k}', t) | p k_{\text{in}} \rangle.$$

The last term gives immediately

$$\langle p' | a_{\text{in}}(k') | p, k_{\text{in}} \rangle = \langle p' k'_{\text{in}} | p k_{\text{in}} \rangle.$$

The main term becomes, using (10)

$$= i \int d^4x \frac{\overleftrightarrow{\partial}}{\partial x_0} \left\{ \langle p' | A(x) | p k_{in} \rangle \frac{\overleftrightarrow{\partial}}{\partial x_0} \frac{\exp[ik'x]}{(2\pi)^{\frac{3}{2}}} \right\} = \\ = i \int d^4x \frac{\exp[ik'x]}{(2\pi)^{\frac{3}{2}}} (\square_x - m^2) \langle p' | A(x) | p k_{in} \rangle ,$$

after partial integrations with respect to the space co-ordinates. Therefore

$$(14) \quad \langle p' k'_{out} | p k_{in} \rangle = \langle p' k'_{in} | p k_{in} \rangle + i \int d^4x \frac{\exp[ik'x]}{(2\pi)^{\frac{3}{2}}} (\square_x - m^2) \langle p' | A(x) | p k_{in} \rangle .$$

This procedure can be continued. By successive use of the asymptotic condition we can «convert» the particles in the state vectors into field operators $A(x)$. In this way we are led in a natural manner to an expression of S in terms of time-ordered or retarded products of field operators. This becomes clear if we start from (14) and take another particle out of the state vectors. The relevant term is

$$(15) \quad \langle p' | A(x) | p k_{in} \rangle = \langle p' | A(x) a_{in}^+(k) | p \rangle = \lim_{y_0 \rightarrow -\infty} \langle p' | A(x) A^+(k, y_0) | p \rangle .$$

To obtain a simple final formula we can proceed in either of two ways:

a) define a time-ordered product by

$$TA(x)A(y) = \begin{cases} A(x)A(y) & \text{if } x_0 > y_0 , \\ A(y)A(x) & \text{if } y_0 > x_0 , \end{cases}$$

then (15) can be written as

$$(16) \quad \lim_{y_0 \rightarrow -\infty} \langle p' | TA(x)A^+(k, y_0) | p \rangle = - \int dy_0 \frac{\partial}{\partial y_0} \langle p' | TA(x)A^+(k, y_0) | p \rangle + \\ + \lim_{y_0 \rightarrow +\infty} \langle p' | TA(x)A^+(k, y_0) | p \rangle .$$

Again, the last term is simple. It gives

$$\lim_{y_0 \rightarrow +\infty} \langle p' | A^+(k, y_0) A(x) | p \rangle = \langle p' | a_{out}^+(k) A(x) | p \rangle = 2k_0 \delta(\mathbf{p}' - \mathbf{k}) \langle 0 | A(x) | p \rangle$$

and this vanishes after application of the Klein-Gordon operator contained in (14). Hence this term does not contribute to the scattering matrix. The other term in (16) leads as before to

$$(17) \quad i \int d^4y \frac{\exp[-iky]}{(2\pi)^{\frac{3}{2}}} (\square_y - m^2) \langle p' | TA(x)A(y) | p \rangle .$$

Hence by (14), (16) and (17)

$$(18) \quad \langle p' k'_{\text{out}} | p k_{\text{in}} \rangle - \langle p' k'_{\text{in}} | p k_{\text{in}} \rangle = \\ = \frac{1}{(2\pi)^3} \int d^4x d^4y \exp [ik'x -iky] (\square_x - m^2)(\square_y - m^2) \cdot \langle p' | T A(x) A(y) | p \rangle .$$

On the other hand, we may

b) define a retarded product by

$$R A(x) A(y) = (-i) \theta(x_0 - y_0) [A(x) A(y)]; \quad \theta(\alpha) = \begin{cases} 1 & \alpha > 0, \\ 0 & \alpha < 0, \end{cases}$$

and get from (15)

$$(15') \quad \langle p' | A(x) a_{\text{in}}^+(k) | p \rangle = \langle p' | A(x), a_{\text{in}}^+(k) | p \rangle + \langle p' | a_{\text{in}}^+(k) A(x) | p \rangle .$$

The last term gives

$$2k_0 \delta(\mathbf{p}' - \mathbf{k}) \langle 0 | A(x) | p' \rangle ,$$

which again vanishes after applying the Klein-Gordon operator of eq. (14). We are left with

$$(16') \quad \langle p' | [A(x), a_{\text{in}}^+(k)] | p \rangle = \lim_{y_0 \rightarrow -\infty} p' [A(x) A^+(\mathbf{k}, y_0)] | p \rangle = \\ = \lim_{y_0 \rightarrow -\infty} i \langle p' | R A(x) A^+(\mathbf{k}, y_0) | p \rangle = -i \int dy_0 \frac{\partial}{\partial y_0} \langle p' | R A(x) A^+(\mathbf{k}, y_0) | p \rangle .$$

There is no boundary term at $y_0 \rightarrow +\infty$, since the retarded product vanishes if $y_0 > x_0$.

Proceeding as before we get from (16')

$$(18') \quad \langle p' k'_{\text{out}} | p k_{\text{in}} \rangle - \langle p' k'_{\text{in}} | p k_{\text{in}} \rangle = \\ = - \frac{i}{(2\pi)^3} \int d^4x d^4y \exp [ik'x -iky] (\square_x - m^2)(\square_y - m^2) \langle p' | R A(x) A(y) | p \rangle .$$

With (18) or (18') we have succeeded in expressing scattering amplitudes in terms of field operators. This is important for our problem because we can now exploit the information we have on these operators—in particular the causality condition—to obtain properties of the amplitudes which follow from our general principles. Using the same methods we can continue to take particles out of the state vectors and finally reduce the scattering matrix to *vacuum expectation values* of appropriate products of field operators.

Moreover, this reduction can be done in a more systematic and elegant manner by establishing some general formulae which may then be applied to transfer particles freely from states to operators and vice versa. I shall only quote some results:

a) General time-ordered products may be introduced by

$$(19) \quad T A(x_1) \dots A(x_n) = A(x_1) \dots A(x_n) \quad \text{if} \quad x_{10} > x_{20} > \dots > x_{n0};$$

$$T A(x_1) \dots A(x_n) \text{ symmetric in } x_1, \dots, x_n.$$

It can then be shown that

$$(20) \quad [S \cdot T A(x_1) \dots A(x_n), A_{\text{in}}(z)] = \int dz' \Delta(z - z') (\square_{z'} - m^2) S T(x_1, \dots, x_n, z').$$

b) General retarded products are defined by

$$(19') \quad R A(x) A(x_1) \dots A(x_n) =$$

$$= (-i)^n \sum \theta(x - x_1) \theta(x_1 - x_2) \dots \theta(x_{n-1} - x_n) [\dots [A(x) A(x_1)] \dots A(x_n)],$$

where the summation is over all permutations of (x_1, \dots, x_n) .

For these products we have the relations

$$(20') \quad [R(x; x_1 \dots x_n), A_{\text{in}}(z)] = -i \int dz' A(z - z') (\square_{z'} - m^2) R(x; x_1 \dots x_n, z').$$

In this manner we obtain for the scattering amplitude of identical particles the following expressions,

$$(18a) \quad \langle p' k'_{\text{out}} | p k_{\text{in}} \rangle - \langle p' k'_{\text{in}} | p k_{\text{in}} \rangle =$$

$$= \frac{1}{(2\pi)^6} \int d^4 x_1 \dots d^4 x_4 \exp [ik' x_1 + ip' x_2 - ik x_3 - ip x_4] \cdot$$

$$\cdot (\square_{x_1} - m^2) \dots (\square_{x_4} - m^2) \langle 0 | T A(x_1) \dots A(x_4) | 0 \rangle,$$

$$(18'a) \quad = \frac{1}{(2\pi)^6} \int d^4 x_1 \dots d^4 x_4 \exp [ik' x_1 + ip' x_2 - ik x_3 - ip x_4] \cdot$$

$$\cdot (\square_{x_1} - m^2) \dots (\square_{x_4} - m^2) \langle 0 | R A(x_1) \dots A(x_4) | 0 \rangle,$$

and similar expressions for more general processes.

In the derivation of dispersion relations the expression (18') has been used mainly. Time-ordered operators seem somewhat more complicated to analyze than retarded operators. However, this may change again.

So far it has not been possible to analyze (18a) or (18'a) completely. The reason is simply that the mathematical problems arising in the case of four-fold operator products have not yet been solved. But I wish to stress that by using these latter relations we may obtain much more information on the scattering amplitude. As long as particles are contained in the state vectors we do not know, loosely speaking, whether they interact in a causal manner; this becomes evident only in the commutation properties of their interpolating fields. Finally, there is no difficulty in writing down corresponding formulae for particles with charge or with spin $\frac{1}{2}$.

To prepare for later applications I rewrite (18'). Introducing the relative co-ordinate $x - y$ we may integrate over the remaining four-vector. Using the notation

$$(21) \quad \langle p' k'_{\text{out}} | p k_{\text{in}} \rangle = \langle p' k'_{\text{in}} | p k_{\text{in}} \rangle + 2\pi i \delta(p + k - p' - k') T(p', k', p, k)$$

and

$$(22) \quad R' A(x) A(y) = (\square_x - m^2)(\square_y - m^2) R A(x) A(y)$$

we obtain easily

$$(23) \quad T = - \int d^4x \exp \left[i \frac{(k + k')}{2} x \right] \langle p' | R' A(x/2) A(-x/2) | p \rangle.$$

We may note that in this formula no assumption has yet been made on the character of p, p' . In particular we may take them as charged scalar particles of mass M (« nucleons ») in contrast to the neutral mass m particles k, k' (« mesons ») and in this way obtain a model for meson-nucleon scattering.

Finally, I derive with the above methods a formula which we shall find useful.

Consider the expression

$$(24) \quad \mathcal{M} = \int d^4x \exp \left[i \frac{(k + k')}{2} x \right] \langle p' | j(x/2) j(-x/2) | p \rangle,$$

where $j(x) = (\square_x - m^2) A(x)$ and the states $|p\rangle, |p'\rangle$ are one-nucleon states; i.e. $p^2 = p'^2 = M^2$. The charged scalar « nucleons » are described by a complex field $\psi(x)$.

The reason for studying the expression (24) is that it is simply related to the imaginary part of the scattering amplitude T .

\mathcal{M} is defined by (24) for arbitrary real vectors k, k' . We restrict ourselves to the case $k + p = k' + p'$.

Inserting on the right hand side of (24) a complete set of intermediate states $|l, \alpha\rangle$ with total energy momentum l and quantum numbers α , we get

$$(25) \quad M = (2\pi)^4 \sum_{\gamma} \langle p' | j(0) | p+k, \gamma \rangle \langle p+k, \gamma | j(0) | p \rangle.$$

Let $\varphi_{\text{in}}(p')$ denote the annihilation operator for an incoming nucleon with momentum p' . Then

$$\begin{aligned} \langle p' | j(0) | p+k, \gamma \rangle &= \langle 0 | \varphi_{\text{in}}(p') j(0) | p+k, \gamma \rangle = \\ &= \langle 0 | [\varphi_{\text{in}}(p'), j(0)] | p+k, \gamma \rangle \quad \text{if } (p+k-p')^2 = k'^2 < m_1^2, \end{aligned}$$

where m_1 is the lowest mass of a state with total momentum k' for which $\langle 0 | j(0) | k' \rangle \neq 0$ (in the π - \mathcal{N} case $m_1 = 3m$).

The second factor in (25) is

$$\langle p+k, \gamma | j(0) | p \rangle = \langle p+k, \gamma | j(0) \varphi_{\text{in}}^+(p) | 0 \rangle = \langle p+k, \gamma | [j(0) \varphi_{\text{in}}^+(p)] | 0 \rangle \quad \text{if } k^2 < m_1^2.$$

We apply now eq. (20') which reads in momentum space

$$(26) \quad [\varphi_{\text{in}}(p'), j(0)] = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^4x \exp[ip'x] R' A(0) \psi(x),$$

to these expressions. If $k^2 < m_1^2$; $k'^2 < m_1^2$ we obtain

$$(27) \quad M = 2\pi \int d^4x_1 d^4x_2 \exp \left[i \frac{(k'-p')}{2} x_1 - i \frac{(k-p)}{2} x_2 \right] \cdot \\ \cdot \sum_{\gamma} \langle 0 | R' A \left(\frac{x_1}{2} \right) \psi \left(-\frac{x_1}{2} \right) | p+k, \gamma \rangle \langle p+k, \gamma | R' A \left(\frac{x_2}{2} \right) \psi^+ \left(-\frac{x_2}{2} \right) | 0 \rangle.$$

2. - Integral representations of causal commutators.

We have seen that S -matrix elements may be expressed in terms of retarded commutators. Next, we determine the mathematical structure of such commutators which follows from local commutativity and the spectrum of the energy-momentum operator.

Consider a matrix element of the commutator of two field operators taken between arbitrary states with total momenta P, Q , which are further specified by quantum numbers α, β ,

$$(28) \quad \langle P, \alpha | [A(x) B(y)] | Q, \beta \rangle = \exp \left[i \left(\frac{P-Q}{2} \right) (x+y) \right] \cdot \\ \cdot \langle P, \alpha | \left[A \left(\frac{x-y}{2} \right) B \left(\frac{y-x}{2} \right) \right] | Q, \beta \rangle.$$

Translation invariance has been used to obtain the right hand side. Apparently, it suffices to study the properties of:

$$(29) \quad \tilde{f}(x) = \langle P, \alpha | \left[A\left(\frac{x}{2}\right) B\left(-\frac{x}{2}\right) \right] | Q, \beta \rangle$$

and its Fourier transform

$$(30) \quad f(q) = \int d^4x \exp[iqx] \tilde{f}(x),$$

$\tilde{f}(x)$ and $f(q)$ must satisfy the following conditions:

$$(31) \quad \tilde{f}(x) = 0 \quad \text{for } x^2 < 0,$$

$$(32) \quad f(q) = 0 \quad \text{unless } \begin{cases} \frac{P+Q}{2} + q \in L^+ \text{ and } \left(\frac{P+Q}{2} + q\right)^2 \geq m_1^2, \\ \text{or} \\ \frac{P+Q}{2} - q \in L^+ \text{ and } \left(\frac{P+Q}{2} - q\right)^2 \geq m_2^2, \end{cases}$$

L^+ denotes the forward light-cone.

m_1 and m_2 are the smallest masses of states such that

$$\langle P, \alpha | A(0) | l_1, \gamma \rangle \neq 0 \quad \text{and} \quad \langle l_1, \gamma | B(0) | Q, \beta \rangle \neq 0$$

or

$$\langle P, \alpha | B(0) | l_2, \gamma \rangle \neq 0 \quad \text{and} \quad \langle l_2, \gamma | A(0) | Q, \beta \rangle \neq 0$$

$$(m_1 = \text{Min } \sqrt{l_1^2}; \quad m_2 = \text{Min } \sqrt{l_2^2}).$$

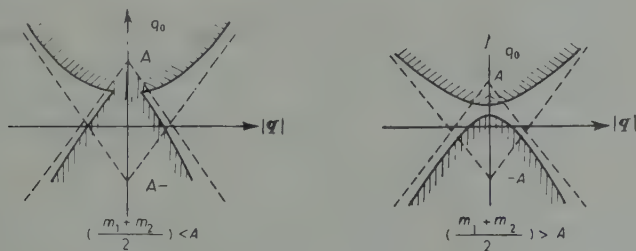
While (31) expresses local commutativity, eq. (32) follows if a sum over intermediate states is introduced in (29). The contribution of the first term of the commutator to $f(q)$ becomes

$$\begin{aligned} \int d^4x \exp[iqx] \langle P, \alpha | A\left(\frac{x}{2}\right) B\left(-\frac{x}{2}\right) | Q, \beta \rangle = \\ = (2\pi)^4 \sum_{\gamma} \langle P, \alpha | A(0) | \frac{P+Q}{2} + q, \gamma \rangle \left\langle \frac{P+Q}{2} + q, \gamma | B(0) | Q, \beta \right\rangle. \end{aligned}$$

This leads—together with an analogous treatment of the second term—directly to (32), which we may call the spectrum condition.

Our problem is to determine the properties of $f(q)$ which arise from (31) and (32). This will be solved by giving an integral representation which expresses all $f(q)$ satisfying (31) and (32) by means of arbitrary weight-functions.

To picture the support of $f(q)$ we choose for a moment a Lorentz system with $\mathbf{P} + \mathbf{Q} = 0$; $(P_0 + Q_0)/2 = A$. $f(q)$ differs from zero inside two hyperboloids which overlap if $(m_1 + m_2)/2 < A$.



(We assume always $A > |m_1 - m_2|/2$, which is satisfied in applications).

I discuss now an Ansatz for $f(q)$ which gives a large class of examples satisfying both requirements. It can then be shown that this Ansatz gives in fact the most general $f(q)$.

Consider

$$(33) \quad \tilde{f}(x) = \int_0^\infty d\kappa^2 \Delta(x, \kappa^2) \tilde{\varphi}(x, \kappa^2),$$

where $\Delta(x, \kappa^2)$ is the usual invariant function with mass κ . This Ansatz satisfies eq. (31) for arbitrary $\varphi(x, \kappa^2)$. On the other hand it seems plausible (and can easily be verified) that any $f(x)$ satisfying (31) can be represented in this way. In Fourier space (33) becomes:

$$(34) \quad f(q) = \int_0^\infty d\kappa^2 \int d^4u \varepsilon(q_0 - u_0) \delta[(q - u)^2 - \kappa^2] \varphi(u, \kappa^2),$$

with $\varphi(u, \kappa^2)$ arbitrary; it is essentially the Fourier transform of $\tilde{\varphi}(x, \kappa^2)$ with respect to x . We now have to satisfy (32).

A point u, κ contributes to $f(q)$ only in points of the hyperboloid

$$(35) \quad (q - u)^2 - \kappa^2 = 0.$$

We call such a hyperboloid admissible if both its sheets lie entirely in that part of q -space where $f(q)$ does not vanish. This determines an admissible region S in u, κ -space such that every point of S gives rise to an admissible

hyperboloid. S is easily computed to consist of the points which satisfy:

$$(36) \quad \left\{ \begin{array}{l} \frac{P+Q}{2} + u \in L^+ \text{ and } \frac{P+Q}{2} - u \in L^+ \\ \text{and} \\ \kappa \geq \text{Max} \left\{ 0; m_1 - \sqrt{\left(\frac{P+Q}{2} + u \right)^2}; m_2 - \sqrt{\left(\frac{P+Q}{2} - u \right)^2} \right\}. \end{array} \right.$$

Taking $\varphi(u, \kappa^2) = 0$ outside S we obtain by (34) functions $f(q)$ satisfying both (31) and (32). The main result due to DYSON [3] is now that the converse is true:

$$(37) \quad \left\{ \begin{array}{l} \text{Any } f(q) \text{ satisfying (31) and (32) has a representation (34) with} \\ \varphi(u, \kappa^2) \neq 0 \text{ only in } S \text{ given by (36).} \end{array} \right.$$

For the proof I refer to Dyson's paper and to Gårding's seminar talk. The latter contains a rigorous mathematical treatment. Actually, it is shown in these proofs that one may restrict the representation to special weight functions $\varphi(u, \kappa^2)$ due to the fact that (34) is not a unique representation. For the applications we have in mind, (34) together with the theorem quoted in (37) is sufficient. Finally, we pass to the retarded commutator given by

$$(38) \quad \begin{aligned} \tilde{F}_R(x) &= \langle P, \alpha | R A \left(\frac{x}{2} \right) B \left(-\frac{x}{2} \right) | Q, \beta \rangle = \\ &= -i\theta(x_0) \langle P, \alpha | \left[A \left(\frac{x}{2} \right) B \left(-\frac{x}{2} \right) \right] | Q, \beta \rangle. \end{aligned}$$

Here we encounter the well-known difficulty that this is in general not a well-defined quantity since a product of distributions is involved. In a formal manner we have ($q' = (q'_0, \mathbf{q})$)

$$F_R(q) = -\frac{1}{2\pi} \int \frac{dq'_0 F(q')}{q'_0 - q_0}; \quad \text{Im } q_0 \geq 0$$

and with (34)

$$(39) \quad F_R(q) = -\frac{1}{2\pi} \int d^4u \int d\kappa^2 \frac{\varphi(u, \kappa^2)}{(q-u)^2 - \kappa^2}.$$

If the integral exists, i.e., if $\varphi(u, \kappa^2)$ is sufficiently bounded in κ^2 , (39) is the desired quantity. In general we may have to divide $\varphi(u, \kappa^2)$ by a polynomial in κ^2 of degree n such that a convergent integral results. Then, as discussed in detail in the quoted literature, $F_R(q)$ is defined up to an arbitrary polynomial in q of degree $2(n-1)$ and may be written as

$$(40) \quad F_R(q) = -\frac{1}{2\pi} \int d^4u \int d\kappa^2 \frac{[(q-u)^2 + m_0^2]^n \varphi(u, \kappa^2)}{[\lambda^2 + m_0^2]^n [(q-u)^2 - \kappa^2]} + R_{2(n-1)}(q),$$

where m_0^2 is arbitrary and R denotes a polynomial in q .

3. - Dispersion relations.

Consider a scattering process such that

$$p + k \rightarrow p' + k',$$

where k, k' are the initial and final momenta of a particle with mass μ , described by a field $A(x)$; p, p' the momenta of a particle of mass M described by a field $\psi(x)$.

Charge and spin values of these particles are not essential for our general problem. For definiteness (and for simplicity) assume that $A(x)$ is a real scalar field (« meson ») and $\psi(x)$ a complex scalar field (« nucleon »). It will be evident how to apply our formulae to other processes, *e.g.* scattering of equal particles ($M = \mu$). Only in the case of production or annihilation of particles is a slight generalization necessary, as then more than two different masses will be needed (in γ - π production: $m_\gamma = 0$ besides μ and M) in processes with two initial and two final particles.

Our aim is to derive as much information on the scattering amplitude as is possible from general principles. To this end we start from the expressions for the scattering amplitude in terms of Heisenberg operators given earlier. As discussed, a formula like (18a) or (18a') which expresses the amplitude as a vacuum expectation value of a product of four operators may be expected to yield more information than expressions where two particles are kept in the state vectors.

So far, however, the mathematical problems connected with products of four operators have not been solved and we must content ourselves with (23) and (27).

The scattering amplitude T is then given by

$$(23) \quad T = - \int d^4x \exp \left[i \frac{(k + k')}{2} x \right] \langle p' | R' A \left(\frac{x}{2} \right) A \left(-\frac{x}{2} \right) | p \rangle$$

as the Fourier transform of a retarded causal commutator. Actually (23) defines T not only on the energy shell ($k^2 = k'^2 = \mu^2$) but (at least) for arbitrary real vectors k, k' . (We keep always the relations $p + k = p' + k'$ and $p^2 = p'^2 = M^2$).

It will be convenient to consider T for the case $k^2 = k'^2 = \zeta$. From invariance under continuous Lorentz transformations it follows then that T depends on three variables. We shall use

$$(41) \quad \left\{ \begin{array}{l} W^2 = (p + k)^2; \quad A^2 = -\frac{(k - k')^2}{4}; \quad \zeta = k^2 = k'^2, \\ \omega = \frac{(k + k')(p + p')}{2\sqrt{(p + p')^2}} \end{array} \right.$$

where

$$W^2 = 2\omega\sqrt{\Delta^2 + M^2} + 2\Delta^2 + M^2 + \zeta,$$

and

$$K^2 = \frac{(W^2 + M^2 - \zeta)^2 - 4M^2W^2}{4W^2},$$

and write $T(\omega, \Delta^2, \zeta)$.

The *scattering amplitude* is given by

$$(42) \quad T(\omega, \Delta^2) = T(\omega, \Delta^2, \mu^2) \quad 0 < \Delta^2 < \infty; \quad \omega > \sqrt{\Delta^2 + \mu^2},$$

where the indicated range of the variables corresponds to the *physical region* i.e. to positive energies of the particles and real scattering angles. 2Δ is the invariant momentum transfer. In the center of mass system W and K are total energy and particle momentum while

$$(43) \quad \cos \vartheta = 1 - \frac{2\Delta^2}{K^2},$$

is the scattering angle (note that $K^2 > \Delta^2$).

The information we have on $T(\omega, \Delta^2, \zeta)$ consists of (23) and (27). To relate the latter equation to T we observe that from (23) it follows that

$$(44) \quad \begin{aligned} \text{Im } T(\omega, \Delta^2, \zeta) &= \frac{1}{2} \int d^4x \exp \left[i \frac{(k+k')}{2} x \right] \langle p' | \left[j\left(\frac{x}{2}\right), j\left(-\frac{x}{2}\right) \right] | p \rangle = \\ &= \frac{1}{2} \{ M(\omega, \Delta^2, \zeta) - M(-\omega, \Delta^2, \zeta) \}. \end{aligned}$$

In writing (44) we have made use of the relation $(k+k')(p-p') = 0$. (The complex conjugate involves an interchange of p and p').

The odd property of $\text{Im } T$ as a function of ω depends on the assumed simple character of the particles described by $j(x)$. If these have other quantum numbers (charge, nucleon number, etc.) the two terms of the commutator will in general refer to different processes with respect to charge, etc. This does not change the mathematical problem.

Our aim is to obtain all properties of $T(\omega, \Delta^2)$ that follow from (23) and (44) in connection with eq. (27) which is valid if $\zeta < m_1^2$, with $m_1 > \mu$ in actual processes. In each case the information is that $T(\omega, \Delta^2, \zeta)$ or its imaginary part is a Fourier transform of a retarded commutator or of a sum over products of such commutators. This we can completely take into account by using Dyson's representation. However, then we have the problem of combining the two representations for T and $\text{Im } T$. This has not been done. Instead a less satisfactory treatment is made where it is not clear that all the information is exploited.

On the other hand one has usually rather definite conjectures as to what properties of the scattering amplitude are to be expected: *i.e.* one looks for *dispersion relations* and this amounts to the following question: Is it true that $T(\omega, \Delta^2)$ is—for fixed Δ^2 —the boundary value of an analytic function which is regular in the upper half ω -plane and is bounded there by a polynomial in ω^2 ? If so, we have a Hilbert relation between the real and imaginary parts of $T(\omega, \Delta^2)$ on the real axis and this is the dispersion relation. Then the further question arises: What statements can be made about the «unphysical region» $\omega < \sqrt{\Delta^2 + \mu^2}$ of a dispersion relation, where T cannot be determined by experiments?

In general, the validity of dispersion relations depends in a complicated manner on the details of the mass spectrum and the value of Δ^2 . This may partially be due to the present rather unsystematic methods used.

In particular cases (and this includes π - \mathcal{N} forward scattering) a dispersion relation follows from (23) alone. The conditions are

1) $\Delta^2 = 0$ (forward scattering).

2) The spectrum condition in (23) is such that matrix elements $\langle p | j(0) | l, \gamma \rangle$ differ from zero, apart from discrete values of l^2 , only for $l^2 > (M + \mu)^2$.

Then also the unphysical region represents no problem. Of actual processes γ -e, π - \mathcal{N} , π - π , π - Ξ scattering satisfy these conditions. (If weaker couplings are neglected.)

In this case it is not necessary to apply our general integral representation to (23). The derivation of the dispersion relation can then be carried through by a more direct and elementary method due to OEHME and SYMANZIK for which I refer to SYMANZIK's paper [4].

Let us turn directly to the general case where in addition eq. (44) is needed. It was first treated by BOGOLJUBOV, MEDVEDEV, POLIVANOV [5]. Then by BREMERMAN, OEHME, TAYLOR [6] who gave a proof under less restrictive assumptions on Δ^2 and on the mass spectrum. Both groups used functions of several complex variables to treat eq. (44).

I shall use Dyson's representation to evaluate (44), which simplifies the proof. For the first part, *i.e.* eq. (23) I follow BOGOLJUBOV *et al.* However, it should also be possible to do this part by Symanzik's method. As already mentioned, none of these treatments including ours is systematic in the sense that it makes use of all the information contained in (23) and (44).

Choose the Lorentz system $\mathbf{p} + \mathbf{p}' = 0$; $p^2 = \Delta^2$ then

$$(45) \quad \begin{cases} p = (\mathbf{p}, \sqrt{\Delta^2 + M^2}); & k = (-\mathbf{p} + \mathbf{e}\sqrt{\omega^2 - \Delta^2} - \zeta, \omega) \\ p' = (-\mathbf{p}, \sqrt{\Delta^2 + M^2}); & k' = (\mathbf{p} + \mathbf{e}\sqrt{\omega^2 - \Delta^2} - \zeta, \omega); \end{cases} \quad \mathbf{e}^2 = 1, \mathbf{e} \cdot \mathbf{p} = 0.$$

$$(46) \quad T(\omega, \Lambda^2, \zeta) = - \int d^4x \exp [i\omega x_0 - i\mathbf{e} \cdot \mathbf{x} \sqrt{\omega^2 - \Lambda^2 - \zeta}] \cdot \langle -\mathbf{p} | R' A \left(\frac{x}{2} \right) A \left(-\frac{x}{2} \right) | \mathbf{p} \rangle ;$$

(46) defines an analytic function $T_1(\omega, \Lambda^2, \zeta)$ of ω and ζ regular in

$$(47) \quad (\mathcal{R}_1): \quad \text{Im } \omega > |\text{Im } \sqrt{\omega^2 - \Lambda^2 - \zeta}|.$$

With $\omega = \omega_1 + i\omega_2$, $\zeta = \zeta_1 + i\zeta_2$ the domain R_1 is explicitly:

$$(47a) \quad \begin{cases} \omega_2 > 0, & \omega_1^2 > \zeta_1 + \Lambda^2, \\ 2\omega_2(\omega_1 - \sqrt{\omega_1^2 - \zeta_1 - \Lambda^2}) < \zeta_2 < 2\omega_2(\omega_1 + \sqrt{\omega_1^2 - \zeta_1 - \Lambda^2}). \end{cases}$$

Take $\zeta = \zeta_1 < -\Lambda^2$. Then T regular in $\text{Im } \omega > 0$, and if T is sufficiently bounded, the relation

$$(48) \quad T(\omega, \Lambda^2, \zeta_1) = \frac{1}{\pi} \int_{-\infty}^{+\infty} d\omega' \frac{\text{Im } T(\omega', \Lambda^2, \zeta_1)}{\omega' - \omega}, \quad \text{Im } \omega > 0,$$

follows. In general we may have to divide T by a polynomial in ω (with no zeros in the upper half ω -plane) to obtain the necessary boundedness. Then, as before, we obtain a generalized relation which involves an arbitrary polynomial in ω . I shall not now treat this in detail. (It seems very plausible on physical grounds that the scattering amplitude cannot increase with a high power of ω at infinity, so that actually the degree of the polynomial is restricted. However so far, it has not been possible to establish such a property from general principles. In the case of one-dimensional scattering SYMANZIK [7] has proved that this is true.)

The real part of (48) is a dispersion relation, however for unphysical (negative) values of ζ . By (44) we have

$$(49) \quad T(\omega, \Lambda^2, \zeta_1) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega' M(\omega', \Lambda^2, \zeta_1) \left\{ \frac{1}{\omega' - \omega} + \frac{1}{\omega' + \omega} \right\}.$$

Introducing W'^2 , related to ω' as in (41), as a new integration variable we obtain:

$$(50) \quad T(\omega, \Lambda^2, \zeta_1) = \frac{1}{2\pi} \int_{M_0^2}^{\infty} dW'^2 M(W'^2, \Lambda^2, \zeta_1) \cdot \left\{ \frac{1}{W'^2 - 2\Lambda^2 - M^2 - \zeta_1 - 2\omega\sqrt{\Lambda^2 + M^2}} + \frac{1}{W'^2 - 2\Lambda^2 - M^2 - \zeta_1 + 2\omega\sqrt{\Lambda^2 + M^2}} \right\}.$$

The lower limit M_0^2 is determined by the fact that W' is equal to the mass of intermediate states in $M(W'^2, A^2, \zeta_1)$ (compare eqs. (25) and (44)). Hence M_0 is the lowest mass ($\sqrt{l^2}$) for which $\langle p | j(0) | l, \gamma \rangle \neq 0$. Frequently, the lowest contribution will come from a single-particle state. (In the $\pi\eta$ case there is a discrete term—the one-nucleon state—at $W' = M$, followed by a continuum which starts at $W' = M + \mu$.)

Following BOGOLJUBOV we show now that a dispersion relation for the scattering amplitude ($\zeta = \mu^2$) follows by analytic continuation of (50), if $M(W'^2, A^2, \zeta)$ has the following analytic property as a function of ζ .

$$(51) \quad \begin{cases} \text{For real } A^2 \geq 0 \text{ and } W'^2 \geq M_0^2, M(W'^2, A^2, \zeta) \text{ is an analytic function} \\ \text{of } \zeta, \text{ regular for } \operatorname{Re} \zeta \leq \mu^2 \text{ in a neighborhood of the real axis. } |\operatorname{Im} \zeta| < \delta. \end{cases}$$

This follows by noting:

1) If (51) holds, then (50) defines an analytic function $T_2(\omega, A^2, \zeta)$ of ω and ζ regular in (R_2) :

$$(\mathcal{R}_2): \quad \zeta_1 \leq \mu^2, \quad |\zeta_2| < \delta, \quad |\zeta_2| < 2\omega_2 \sqrt{A^2 + M^2}.$$

For ζ real and $\zeta < -A^2$, $T_1(\omega, A^2, \zeta) = T_2(\omega, A^2, \zeta)$.

$$2) \quad \lim_{\omega_1 \rightarrow 0} T_2(\omega, \zeta_1, A^2) = \lim_{\substack{\omega_2 \rightarrow 0 \\ \zeta_2(\omega_2) \rightarrow 0}} T_2(\omega, \zeta, A) \quad \text{if}$$

$$\text{if} \quad |\zeta_2(\omega_2)| < \min \{ \delta, 2\omega_2 \sqrt{A^2 + M^2} \}; \quad \zeta_1 \leq \mu^2.$$

3) The boundary value of T_2 for $\zeta = \mu^2$, $\omega_2 \rightarrow 0$, $\omega_1 > \sqrt{A^2 + \mu^2}$ is the physical scattering amplitude $T(\omega; A^2)$ defined by (46). This is true because this boundary can be reached along a path which lies in the intersection of R_1 and R_2 .

Take $\omega_1 > \sqrt{A^2 + \mu^2}$, $\omega_2 > 0$, $\zeta_1 \leq \mu^2$

$$(52) \quad \begin{cases} |\zeta_2| < \operatorname{Min} \{ \delta, 2\omega_2 \sqrt{A^2 + M^2} \}, \\ 2\omega_2(\omega_1 - \sqrt{\omega_1^2 - \zeta_1^2 - A^2}) < \zeta_2 < 2\omega_2(\omega_1 + \sqrt{\omega_1^2 - \zeta_1^2 - A^2}). \end{cases}$$

This allows a finite interval for $\zeta_2(\omega_2)$ in which to approach the boundary.

Analytic properties of $M(W^2, A^2, \zeta)$. — Our remaining problem is to establish the analytic properties of M as needed in (51).

Let us go back to eq. (27) and introduce Dyson's representation for the retarded commutators appearing in this equation.

The corresponding unretarded commutators are

$$(53) \quad \int d^4x_1 d^4x_2 \exp \left[i \frac{k' - p'}{2} x_1 - i \frac{k - p}{2} x_2 \right] \sum_{\gamma} \langle 0 | \left[j \left(\frac{x_1}{2} \right) f \left(\frac{-x_1}{2} \right) | p + k, \gamma \rangle \cdot \right. \\ \left. \cdot \langle p + k, \gamma' | \left[j \left(\frac{x_2}{2} \right) f^* \left(\frac{-x_2}{2} \right) | 0 \rangle \right], \right.$$

where $f(x) = (\square_x - M^2) \psi(x)$.

To each commutator in (53) corresponds a weight function $\varphi_{\gamma}(u, \kappa^2, p+k)$ which vanishes unless

$$(54) \quad \begin{cases} \frac{p+k}{2} + u \in L^+; & \frac{p+k}{2} - u \in L^+, \\ \kappa > \text{Max} \left\{ 0; m_1 - \sqrt{\left(\frac{p+k}{2} + u \right)^2}; m_2 - \sqrt{\left(\frac{p+k}{2} - u \right)^2} \right\}; \end{cases}$$

where m_1 and m_2 are determined as in (32). (For the $\pi\mathcal{N}$ case, $m_1 = 3\mu$, $m_2 = M + \mu$.)

These individual functions φ_{γ} are then multiplied and added according to (53) to give an overall function,

$$(55) \quad \varphi(u_1, \kappa_1, u_2, \kappa_2; p+k) = \sum_{\gamma} \varphi_{\gamma}(u_1, \kappa_1, p+k) \varphi_{\gamma}^*(u_2, \kappa_2, p+k),$$

which satisfies the support condition (54) in each pair of variables u, κ separately and is a real, invariant function of the four-vectors $u_1, u_2, p+k$ (we may restrict φ to be real by taking the real part of (55); (an imaginary part does not contribute to the *real* quantity M which we represent). Also φ has to satisfy a positive definiteness condition, as follows from (55). However, we shall not use this property which is related to unitarity.

The representation for $M(W^2, \Delta^2, \zeta)$ follows then from (53) by going over to retarded commutators,

$$(56) \quad M(W^2, \Delta^2, \zeta) = \frac{1}{2\pi} \int \frac{d^4u_1 d^4u_2 d\kappa_1^2 d\kappa_2^2 \varphi(u_1, \kappa_1, u_2, \kappa_2, p+k)}{[(k' - p)/2 - u_1]^2 - \kappa_1^2 [(k - p)/2 - u_2]^2 - \kappa_2^2}.$$

In general, M will not be bounded enough for (56) to hold in this form. However, the introduction of polynomials (as in (40)) does not change the analytic properties we are interested in and for this reason I continue to discuss (56) as it stands.

To evaluate (56) take the center of mass system, $\mathbf{p} + \mathbf{k} = 0$. Then

$$(57) \quad M(W^2, A^2, \zeta) = \\ = \frac{1}{2\pi} \int \left[\left(\frac{M^2 - \zeta}{2W} + u_{10} \right)^2 - \kappa_1^2 - (\mathbf{k}' - \mathbf{u}_1)^2 \right] \left[\left(\frac{M^2 - \zeta}{2W} + u_{20} \right)^2 - \kappa_2^2 - (\mathbf{k} - \mathbf{u}_2)^2 \right] \\ \cdot d^4 u_1 d^4 u_2 d\kappa_1^2 d\kappa_2^2 \varphi(u_{10}, \mathbf{u}_1^2, \kappa_1^2, \mathbf{u}_1 \mathbf{u}_2 / (|\mathbf{u}_1| |\mathbf{u}_2|), W)$$

Further, introduce polar co-ordinates such that

$$\mathbf{k} = K(1, 0, 0); \quad \mathbf{u}_1 = u_1(\cos \varphi_1 \sin \vartheta_1, \sin \varphi_1 \sin \vartheta_1, \cos \vartheta_1)$$

$$\mathbf{k} = K(\cos \vartheta, \sin \vartheta, 0); \quad \mathbf{u}_2 = u_2(\cos \varphi_2 \sin \vartheta_2, \sin \varphi_2 \sin \vartheta_2, \cos \vartheta_2)$$

and use $\chi = \varphi_1$ and $\alpha = \varphi_1 - \varphi_2$ as integration variables,

$$M = \frac{1}{8\pi K^2} \cdot \\ \cdot \frac{\int du_{i0} u_i du_i d\kappa_i \int_0^{2\pi} d\alpha \int_0^{2\pi} d\chi \int_0^\pi d\vartheta_i \varphi(u_{i0}, u_i^2, \kappa_i^2, \cos \alpha \sin \vartheta_1 \sin \vartheta_2 + \cos \vartheta_1 \cos \vartheta_2, W)}{[x_1(\zeta) - \cos(\vartheta - \chi)][x_2(\zeta) - \cos(\chi - \alpha)]},$$

where

$$x_i(\zeta) = \frac{K^2 + u_i^2 + \kappa_i^2 - (u_{i0} + ((M^2 - \zeta)/2W))^2}{2Ku_i \sin \vartheta_i}, \\ K^2 = \frac{(W^2 + M^2 - \zeta)^2 - 4M^2 W^2}{4W^2}; \quad \cos \vartheta = 1 - \frac{2A^2}{K^2}.$$

Next, we may carry out some of the integrations in (58). By a change of variables

$$u'_i = u_i \sin \vartheta_i, \\ u'^2_i + \kappa'^2 = u_i^2 + \kappa_i^2,$$

we eliminate ϑ_i from the denominators. These variables remain only in φ and their integration results merely in a new weight function $\varphi(u_{i0}, u_i, \kappa_i, \cos \alpha, W)$. The range of u' , κ' is again given by (54). Moreover, we integrate over χ with the aid of

$$(59) \quad \int_0^{2\pi} d\chi \frac{1}{[x_1 - \cos(\vartheta - \chi)]} \cdot \frac{1}{[x_2 - \cos(\chi - \alpha)]} = \\ = \frac{2\pi}{x_1 x_2 \sqrt{x_1^2 - 1} \sqrt{x_2^2 - 1} - \cos(\vartheta - \alpha)} \cdot \frac{x_1/(\sqrt{x_1^2 - 1}) + x_2/(\sqrt{x_2^2 - 1})}{x_1 x_2 \sqrt{x_1^2 - 1} \sqrt{x_2^2 - 1} - \cos(\vartheta - \alpha)}.$$

Hence,

$$(60) \quad M(W, \Delta^2, \zeta) = \int du_{i0} du_i d\kappa_i^2 \int_0^{2\pi} d\alpha \bar{\varphi}(u_{i0}, u_i, \kappa_i^2, \cos \alpha, W) \cdot \frac{[y_1/(\sqrt{y_1^2 - K^2}) + y_2/(\sqrt{y_2^2 - K^2})]}{y_1 y_2 + \sqrt{y_1^2 - K^2} \sqrt{y_2^2 - K^2} - K^2 \cos(\vartheta - \alpha)},$$

with

$$y_i = \frac{K^2 + u_i^2 + \kappa_i^2 - (u_{i0} + (M^2 - \zeta)/2W)^2}{2u_i}.$$

As follows from (54) the variables u_{i0} , u_i , κ_i vary in

$$(61) \quad \begin{cases} 0 \leq u \leq \frac{W}{2}; & |u_0| \leq \frac{W}{2} - u, \\ \kappa \geq \text{Max} \left\{ 0; m_1 - \sqrt{\left(\frac{W}{2} + u_0\right)^2 - u^2}; m_2 - \sqrt{\left(\frac{W}{2} - u_0\right)^2 - u^2} \right\}. \end{cases}$$

From (60) we may expect that M has analytic properties both in ζ and Δ^2 since the dependence on these variables is completely contained in the second factor. We may restrict ourselves to real $\Delta^2 > 0$ and essentially also to real $\zeta \leq \mu^2$, since we need only a neighborhood of the real ζ -axis.

According to (60) the function $M(W^2, \Delta^2, \zeta)$ will have singularities if

$$(62) \quad \begin{cases} 1) & y - K^2 = 0, \\ 2) & y_1 y_2 + \sqrt{y_1^2 - K^2} \sqrt{y_2^2 - K^2} - K^2 \cos(\vartheta - \alpha) = 0. \end{cases}$$

We have to discuss whether the points $\zeta \leq \mu^2$ on the real axis are regular points.

For real ζ , y_i , K^2 (which may be negative) and $\cos \vartheta$ are also real. Calculate first the minimum value of $y_i^2 - K^2$ if u_{i0} , u_i , κ_i^2 vary over the region (61) and $\zeta \leq \mu^2$. The result is for $K^2 \geq 0$

$$(63) \quad \text{Min}\{y_i^2 - K^2\} = \frac{(m_1^2 - \mu^2)(m_2^2 - M^2)}{W^2 - (m_1 - m_2)^2}.$$

Therefore the first singularity in (62) does not occur if $m_1 > \mu$, $m_2 > M$, $W^2 > (m_1 - m_2)^2$. This is satisfied in all applications.

Consider now the second singularity.

Solving for $\cos \vartheta$ we get

$$(64) \quad K^2 \cos \vartheta = Y \cos \alpha \pm i \sqrt{Y^2 - K^4} \sin \alpha,$$

where

$$Y = y_1 y_2 + \sqrt{y_1^2 - K^2} \sqrt{y_2^2 - K^2}.$$

If $K^2 > 0$ we have shown already that $y_i^2 > K^2$ and in fact $y_i > 0$ as is easily checked. We shall see in a moment that a necessary condition for our proof is that $y_i > 0$ also for $K^2 < 0$.

If this is satisfied, then $Y > 0$, $Y^2 > K^4$ and (64) can be fulfilled only for $\sin \alpha = 0$.

Then singularities can occur if

$$K^2 \cos \vartheta = K^2 \left(1 - \frac{2\Delta^2}{K^2} \right) = \pm Y.$$

Because of $Y > |K^2|$ only the minus sign is possible. Then (65) can be written:

$$(66) \quad \Delta^2 = \frac{1}{2}(K^2 + Y) = \frac{1}{2}\{K^2 + y_1 y_2 + \sqrt{y_1^2 - K^2} \sqrt{y_2^2 - K^2}\}.$$

The values which the right-hand side of this relation takes on if the parameters vary will now determine the allowed values of Δ^2 . The minimum value of the r.h.s. (its maximum value is ∞) determines the maximum value of Δ^2 for which the dispersion relation can be proved. If $y_i = 0$ is possible the r.h.s. vanishes and we cannot, even for forward scattering obtain a dispersion relation.

We have

$$\begin{aligned} \Delta^2 &= \frac{1}{2} \text{Min} \{K^2 + y_1 y_2 + \sqrt{y_1^2 - K^2} \sqrt{y_2^2 - K^2}\} = \\ &= \frac{1}{2} \text{Min} \{K^2 + 2y^2 - K^2\} = \text{Min} (y^2), \end{aligned}$$

this gives

$$(67) \quad \Delta^2 = K^2 + \frac{(m_1^2 - \mu^2)(m_2^2 - M^2)}{W^2 - (m_1 - m_2)^2};$$

where

$$K^2 = \frac{[W^2 - (m + \mu)^2][W^2 - (m - \mu)^2]}{4W^2}.$$

From this relation we can calculate the maximum allowed value of Δ^2 by taking the minimum of the r.h.s. with respect to W . The result depends of

course on the process we consider, *i.e.* on m_1 , m_2 and the range of W . For $\pi\text{-}\mathcal{N}$ scattering:

$$m_1 = 3\mu, \quad m_2 = M + \mu, \quad W \geq M + \mu$$

(where μ and M are pion and nucleon mass) the result is

$$(68) \quad \Delta_{\max}^2 = \frac{8}{3} \frac{2M + \mu}{2M - \mu} \mu^2.$$

As another example, consider $\mathcal{N}\text{-}\mathcal{N}$ scattering. In this case the relevant masses are easily shown to be $\mu = M$ (nucleon mass), $m_1 = m_2 = M + m_\pi$ and $W \geq 2m_\pi$. (67) leads to

$$(69) \quad \Delta^2 = \frac{W^2}{4} - M^2 + \frac{m_\pi^2(2M + m_\pi)^2}{W^2}$$

with the minimum

$$(70) \quad \Delta_{\max}^2 = m_\pi(2M + m_\pi) - M^2.$$

This is positive only if

$$(71) \quad \frac{m_\pi}{M} > \sqrt{2} - 1,$$

and only if the ratio m_π/M satisfied this condition could we prove a dispersion relation for forward $\mathcal{N}\text{-}\mathcal{N}$ scattering.

Our discussion is still incomplete in several respects.

1) $M(W^2, \Delta^2, \zeta)$ will in general have contributions from one-particle states as is evident from (25) or (27). It is convenient to separate these terms from the continuum, since they are of a particularly simple form.

Take our model of $\pi\text{-}\mathcal{N}$ scattering. There is a contribution from the one-nucleon state with $W^2 = M^2$. Write

$$(72) \quad M(W, \Delta^2, \zeta) = M_1(W, \Delta^2, \zeta) + M_2(W, \Delta^2, \zeta),$$

where M_1 is the one-nucleon contribution. Actually, this does not depend on Δ^2 since it is a single term in the sum over states in (25). (72) induces a corresponding division of the weight function φ into two parts, $\varphi = \varphi_1 + \varphi_2$, where according to (55) φ_1 is of the simple form

$$(73) \quad \varphi_1(u_1, \kappa_1, u_2, \kappa_2, p+k) = \varphi_1(u_1, \kappa_1, p+k) \varphi_1^*(u_2, \kappa_2, p+k).$$

This term is very easy to treat since it does not depend on the scalar product $u_1 u_2$.

Instead of (60) we obtain for M_1 simply

$$(74) \quad M_1(W^2, \zeta) = \int \frac{du_{i0} du_0 d\kappa_i^2 \varphi(u_{i0}, u_1^2, \kappa_1^2, W) \varphi_{12}(u_{20}, u_2^2, \kappa_2^2, W)}{[y_1^2 - K^2][y_2^2 - K^2]}.$$

We know already that $y^2 > K^2$ so that there are no singularities for $\zeta < \mu^2$. On the energy shell ($\zeta = \mu^2$) $M_1(W^2, \mu^2)$ becomes simply

$$M_1(W^2, \mu^2) = g^2 \cdot \delta(W^2 - M^2),$$

where g^2 is (apart from constant factors) a conventional coupling constant.

2) In non-forward scattering we always encounter a non-physical region in the dispersion relation integral corresponding to values $\omega' < \sqrt{\Delta^2 + \mu^2}$ in (49) or $W'^2 < (\sqrt{\Delta^2 + M^2} + \sqrt{\Delta^2 + \mu^2})^2$ in (50) where the integrand $M(W, \Delta^2) = 2 \operatorname{Im} T(W, \Delta^2)$ has not yet been expressed by experimental quantities. This can be done [8] if—as in $\pi\mathcal{N}$ scattering—there is no unphysical region in forward scattering, *i.e.* we have for the contribution from the continuum of states $W'^2 \geq (M + \mu)^2$.

Consider eq. (60) on the energy shell $\zeta = \mu^2$. We are now interested in the dependence on Δ^2 . Since y_i and K^2 do not depend on Δ^2 we may simply introduce

$$z = \frac{y_1 y_2 + \sqrt{y_1^2 - K^2} \sqrt{y_2^2 - K^2}}{K^2}$$

as a new integration variable. Then

$$(76) \quad M(W, \Delta^2) = \int_{z_0}^{\infty} dz \int_0^{2\pi} d\alpha \frac{\Phi(z_1 \cos \alpha, W)}{z - \cos(\vartheta - \alpha)},$$

where

$$z_0 = 1 + \frac{2(m_1^2 - \mu^2)(m_2^2 - m^2)}{K^2 [W^2 - (m_1 - m_2)^2]}.$$

From (76) it is clear that $M(W, \Delta^2)$ —which is the imaginary part of the scattering amplitude—has analytic properties as a function of Δ^2 or $\cos \vartheta = 1 - (2\Delta^2/K^2)$,

$M(W, \Delta^2)$ is an analytic function of $\cos \vartheta$, regular in an ellipse with foci at ± 1 and with semi-major axis z_0 . This includes the physical region $-1 \leq \cos \vartheta \leq +1$.

Then the partial wave expansion of $\text{Im } T = 2M(W, \Delta^2)$

$$(77) \quad \text{Im } T(W^2, \Delta^2) = \frac{1}{\pi^2} \frac{W}{K} \sum_{l=0}^{\infty} (2l+1) \text{Im } C_l(W) P_l \left(1 - \frac{2\Delta^2}{K^2} \right),$$

converges inside the ellipse. In this way we may in principle first obtain the coefficients $C_l(W)$ from the physical amplitude and then define the unphysical region by (77). It is easily checked that this definition is valid in the entire interval.

3) Collecting these results we may finally write the physical dispersion relation as it follows from (50):

$$(78) \quad \text{Re } T(W, \Delta^2) = g^2 \left[\frac{1}{W^2 - 4\Delta^2 - M^2 - 2\mu^2} - \frac{1}{W^2 - M^2} \right] + \\ + \frac{1}{\pi} P \int_{(M+\mu^2)}^{\infty} dW'^2 \text{Im } T(W', \Delta^2) \cdot \left\{ \frac{1}{W'^2 - W^2} + \frac{1}{W'^2 - 4\Delta^2 - 2(M^2 + \mu^2) + W^2} \right\}.$$

As mentioned before a generalized relation is necessary and is derived under the same conditions as before, if the scattering amplitude is not sufficiently bounded. For these cases I refer to the quoted literature.

4) We have not considered the vertex function and two-particle production amplitudes (γ - π production).

Several authors have investigated the vertex function. The case of γ - π production was treated recently by OEHME and TAYLOR.

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Theory of Coupled Quantized Fields.

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We consider a field theory in which the equations of motion are known and we look for the algebraic tools to handle these equations. The particular theory to which we shall refer is electrodynamics.

There are two algorithms, which we must consider: the pfaffians and the hafnians. The pfaffians, well known since a long time, appear in a natural way when we consider fermion fields. Correspondingly, when we consider a boson field the appropriate algebraic tool is the hafnian which has been introduced for the first time in connection with this kind of problems.

Let x_1, x_2, \dots, x_n be the elements of an algebra. For them we define two different kinds of product

(Grassmann algebra)

$$(1) \quad x_h \wedge x_k = -x_k \wedge x_h \quad (x_h \wedge x_h = 0);$$

(Clifford Algebra)

$$(2) \quad x_h \wedge x_k = -x_k \wedge x_h + 2\delta_{hk} \quad (x_h \wedge x_h = 1).$$

1. - Determinants.

Let

$$(3) \quad \omega_h = \sum_{i=1}^h \alpha_{hi} x_i$$

be a linear form in the elements x_i ($i = 1, 2, \dots, n$).

We are interested in finding a convenient expression of products like:

$$\omega_1 \wedge \omega_2.$$

We have by virtue of (1):

$$\omega_1 \wedge \omega_2 = \sum_{i \neq j} \alpha_{1i} \alpha_{2j} x_i \wedge x_j = \sum_{i < j} \begin{vmatrix} \alpha_{1i} & \alpha_{1j} \\ \alpha_{2i} & \alpha_{2j} \end{vmatrix} x_i \wedge x_j.$$

We see that determinants have been introduced in a quite natural way.

In the general case of products like:

$$\Pi = \omega_1 \wedge \omega_2 \dots \wedge \omega_k,$$

we have, by an easy extension, using the Sylvester's notation for determinants, *i.e.*:

$$(4) \quad \begin{pmatrix} 1 & 2 & \dots & k \\ i_1 & i_2 & \dots & i_k \end{pmatrix} = \begin{vmatrix} \alpha_{1i_1} & \alpha_{1i_2} & \dots & \alpha_{1i_k} \\ \alpha_{2i_1} & \alpha_{2i_2} & \dots & \alpha_{2i_k} \\ \dots & \dots & \dots & \dots \\ \alpha_{ki_1} & \alpha_{ki_2} & \dots & \alpha_{ki_k} \end{vmatrix}$$

$$\begin{aligned} \Pi &= \sum_{i_1 < i_2 \dots i_k} \begin{pmatrix} 1 & 2 & \dots & k \\ i_1 & i_2 & \dots & i_k \end{pmatrix} x_{i_1} \wedge x_{i_2} \wedge \dots \wedge x_{i_k} = \\ &= \frac{1}{k!} \sum_{i_1, i_2, \dots, i_k} \begin{pmatrix} 1 & 2 & \dots & k \\ i_1 & i_2 & \dots & i_k \end{pmatrix} x_{i_1} \wedge x_{i_2} \wedge \dots \wedge x_{i_k}. \end{aligned}$$

If $k = n$ there is only one permutation $i_1 = 1, i_2 = 2, \dots, i_n = n$ possible and we get:

$$(5) \quad \omega_1 \wedge \omega_2, \dots \wedge \omega_n = \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix} x_1 \wedge x_2 \wedge \dots \wedge x_n.$$

Eq. (3), (4), (5) could be used in order to define determinants, and all theorems about determinants could be deduced immediately by them.

For instance, one could prove the following expansion of determinants by minors of the rows 1 and 2:

$$\begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix} = \sum_{\substack{i_1 < i_2 \\ i_3 < \dots < i_n}} (-1)^p \begin{pmatrix} 1 & 2 \\ i_1 & i_2 \end{pmatrix} \begin{pmatrix} 3 & \dots & n \\ i_3 & \dots & i_n \end{pmatrix},$$

where p is the parity of i_1, i_2, \dots, i_n with respect to their natural order 1, 2, ..., n .

We have:

$$\begin{aligned}\omega_1 \wedge \omega_2 \dots \wedge \omega_n &= (\omega_1 \wedge \omega_2) \wedge (\omega_3 \wedge \dots \wedge \omega_n) = \\ &= \left[\sum_{i_1 < i_2} \begin{pmatrix} 1 & 2 \\ i_1 & i_2 \end{pmatrix} x_{i_1} \wedge x_{i_2} \right] \wedge \left[\sum_{i_3 < \dots < i_n} \begin{pmatrix} 3 & \dots & n \\ i_3 & \dots & i_n \end{pmatrix} x_{i_3} \wedge \dots \wedge x_{i_n} \right] = \\ &= \sum_{\substack{i_1 < i_2 \\ i_3 < \dots < i_n}} (-1)^p \begin{pmatrix} 1 & 2 \\ i_1 & i_2 \end{pmatrix} \begin{pmatrix} 3 & \dots & n \\ i_3 & \dots & i_n \end{pmatrix} x_1 \wedge x_2 \dots \wedge x_n.\end{aligned}$$

By comparing the last expression with eq. (5) we get the result.

2. - Pfaffians.

The pfaffians can be introduced in several ways: the most useful are the following:

Start from a quadratic external form:

$$\Omega = \frac{1}{2!} \sum_{h, k=1}^n \alpha_{hk} x_h \wedge x_k = \sum_{h < k} \alpha_{hk} x_h \wedge x_k,$$

with:

$$\alpha_{hk} = -\alpha_{kh}.$$

Its (Grassmann) powers are:

$$\Omega \wedge \Omega \dots \wedge \Omega = (\Omega)^i \wedge.$$

We consider $(\Omega)^2 \wedge$:

$$\begin{aligned}\Omega \wedge \Omega &= \sum_{h_1 < h_2} \sum_{h_3 < h_4} \alpha_{h_1 h_3} \alpha_{h_2 h_4} x_{h_1} \wedge x_{h_2} \wedge x_{h_3} \wedge x_{h_4} = \\ &= 2! \sum_{h_1 < h_2 < h_3 < h_4} (\alpha_{h_1 h_3} \alpha_{h_2 h_4} - \alpha_{h_1 h_4} \alpha_{h_2 h_3} + \alpha_{h_1 h_4} \alpha_{h_2 h_3}) x_{h_1} \wedge x_{h_2} \wedge x_{h_3} \wedge x_{h_4}.\end{aligned}$$

From now on, when $h < k$ we shall use the notation $\alpha_{hk} = (hk)$.

Therefore:

$$\begin{aligned}\Omega \wedge \Omega &= 2! \sum_{h_1 < h_2 < h_3 < h_4} [(h_1 h_2)(h_3 h_4) - (h_1 h_3)(h_2 h_4) + (h_1 h_4)(h_2 h_3)] x_{h_1} \wedge \dots \wedge x_{h_4} \\ &= 2! \sum_{h_1 < h_2 < h_3 < h_4} (h_1 h_2 h_3 h_4) x_{h_1} \wedge x_{h_2} \wedge x_{h_3} \wedge x_{h_4},\end{aligned}$$

where we have introduced the symbol:

(which we shall call *pfaffian*): $(1 \ 2 \ 3 \ 4) = (1 \ 2)(3 \ 4) - (1 \ 3)(2 \ 4) + (1 \ 4)(2 \ 3).$

Therefore in the general case the definition of a pfaffian will be given by:

$$(6) \quad (\Omega)_{\wedge}^i = i! \sum_{h_1 > h_2 \dots > h_{2i}} (h_1 h_2 \dots h_{2i}) x_{h_1} \wedge x_{h_2} \dots x_{h_{2i}}.$$

For $n = 2m$ the maximum value of i is m , because the Grassmann product of two equal elements is zero.

For the same reason when $n = 2m + 1$ the maximum value of i is m .

In order to find the expansion rule for pfaffians we apply the definition (6) and we get in complete analogy with the case of determinants:

$$(7) \quad (1 \ 2 \dots 2m) = \sum_{h=2}^{2m} (-1)^h (1 \ h)(2 \dots h-1, h+1 \dots 2m).$$

At this point it is convenient to represent a pfaffian as a triangular array, that is:

$$(1 \ 2 \dots 2m) = \left| \begin{array}{cccc} (1 \ 2) & (1 \ 3) & \dots & (1 \ 2n) \\ & (2 \ 3) & \dots & (2 \ 2n) \\ & & \ddots & \vdots \\ & & & (2n-1, 2n) \end{array} \right|.$$

In fact a pfaffian (in analogy with a determinant) can be expanded by the elements of one of its lines: « line h » is that one, in the triangular array, which contains all the elements carrying the index h , regardless of whether h is the first or the second index. The element (hk) belongs to the lines h and k ; crossing these lines out, what is left is a new pfaffian, the « minor » of (hk) . The rule for the expansion is then: « A pfaffian is equal to the sum of the product of each element (hk) of a prefixed line by its minor, each term in the sum being given the sign $(-1)^{h+k+1}$ ». For instance if we fix the line 1, this rule leads to eq. (7).

By repeated application of eq. (7) we obtain:

$$(1 \ 2 \dots 2m) = \sum' (-1)^p (i_1 \ i_2)(i_3 \ i_4) \dots (i_{2n-1} \ i_{2n}).$$

\sum' means here, and *throughout the following*, summation over all the permutations $i_1 \ i_2 \dots i_{2n}$ of $1 \ 2 \dots 2n$ which satisfy the limitations:

$$i_1 < i_2 \quad i_3 < i_4 \quad \dots \quad i_{2n-1} < i_{2n} \quad i_1 < i_3 < i_5 < \dots < i_{2n-1}.$$

A fundamental theorem on pfaffians (that we will not prove) is the

following:

$$\begin{vmatrix} 0 & (1\ 2) & (1\ 3) & \dots & (1, 2m) \\ -(1\ 2) & 0 & (2\ 3) & & . \\ -(1\ 3) & -(2\ 3) & 0 & & . \\ . & . & . & . & . \\ -(1, 2m) & . & . & . & 0 \end{vmatrix} = (1\ 2 \dots 2m)^2.$$

Exercise 1. Show that if $(hk) = -(kh)$ then $(1\ 2 \dots 2m)$ is antisymmetrical under permutations.

Exercise 2. Show that:

$$(1\ 2 \dots 2m) = \frac{(-1)^{m/2}}{2^m} \sum (-1)^p \begin{pmatrix} h_1 h_2 \dots h_m \\ k_1 k_2 \dots k_m \end{pmatrix},$$

where p is the parity of the permutation $k_1 k_2 \dots k_m$, $h_1 h_2 \dots h_m$ of $1\ 2 \dots 2m$ and \sum is the sum over all the $\binom{2m}{m}$ permutations such that:

$$k_1 < k_2 \dots < k_m; \quad h_1 < h_2 \dots < h_m.$$

Exercise 3. Consider a product of 4×4 matrices:

$$P_1 P_1 \dots P_{2m},$$

where $P_r = \gamma^\mu p_\mu^{(r)}$ and $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\delta^{\mu\nu}$.

Show that:

$$\frac{1}{4} \text{Sp}(P_1 P_2 \dots P_{2m}) = (1\ 2 \dots 2m),$$

where (hk) is the scalar product $\mathbf{p}^{(h)} \mathbf{p}^{(k)}$.

If we count how many terms there are in a pfaffian we find: number of terms of

$$(1\ 2) \dots 2m) = (2m-1)!! = \frac{(2m)!}{2^m m!}.$$

Now we state (without proof) the main theorem about pfaffians. This is an extension of what we know about determinants. This theorem which contains as a particular case eq. (7) is expressed by the following identity:

$$(7-bis) \quad (1\ 2 \dots 2m) = \sum_{r=0}^{[q/2]} (-1)^{\binom{q-2r}{2}} \sum_{c_r} (-1)^p (k'_1 k'_2 \dots k'_{2r}) \cdot \begin{pmatrix} k''_1 & k''_2 & \dots & k''_s \\ l''_1 & l''_2 & \dots & l''_s \end{pmatrix} (l'_1 l'_2 \dots l'_{2s}),$$

where the expansion is made with respect to $q = 2r + s$ indices $k_1 k_2 \dots k_q$ arbitrarily chosen in the pfaffian.

(Of course $s + 2t + q = 2m$)

\sum_{c_r} is extended to all the $\binom{q}{2r} \binom{2m-q}{2t}$ permutations:

$$k'_1 < k'_2 \dots < k'_{2r} ; \quad k''_1 < k''_2 < \dots < k''_s$$

of the indices $k_1 k_2 \dots k_q$, and

$$l''_1 < l''_2 \dots < l''_s ; \quad l'_1 < l'_2 \dots < l'_{2t}$$

of the indices $l_1 l_2 \dots l_{2m-q}$.

3. - Relation between Grassmann's algebra and Clifford's algebra.

The relation between the two algebras is given by the following formula which we do not prove:

$$(8) \quad \omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_k = \sum_{r=0}^{[k/2]} \sum_{c_r} (-1)^p (h_1 h_2 \dots h_{2r}) \omega_{i_1} \wedge \omega_{i_2} \dots \omega_{i_{k-2r}} ;$$

where $(h_1 h_2) = \sum_i \alpha_{h_1 i} \alpha_{h_2 i}$, etc.:

$$h_1 < h_2 \dots < h_{2r} ; \quad l_1 < l_2 < \dots < l_{k-2r}$$

is a permutation of $1 \ 2 \dots k$ with parity p ;

\sum_{c_r} is extended over all such permutations with fixed r . $[k/2]$ is the highest integer contained in $k/2$. In the case $k=2$ eq. (8) gives:

$$\omega_1 \wedge \omega_2 = (1 \ 2) + \omega_1 \wedge \omega_2 .$$

This formula is essentially analogous to Wick's theorem on field operators:

$$T(AB) = :AB: + N(AB) ,$$

where $:AB:$ represents the contracted factors.

The Clifford product that we have defined is just the kind of product which appears when we consider creation and annihilation operators of the Fermion field. Therefore determinants and pfaffians are the appropriate mathematical tools for the Fermion field.

We can define forms in the elements $x_1 x_2 \dots x_h$ with the rules:

$$x_h \amalg x_k = \begin{cases} x_k \amalg x_h & h \neq k \\ 0 & h = k \end{cases}$$

and

$$x_h \amalg x_k = x_k \amalg x_h \quad h \neq k$$

$$x_h \amalg x_h = 1$$

which corresponds to Grassmann product and Clifford product respectively.

Entirely similar results can be obtained from these commutation rules. And the only difference will be, in all the cases, the change of the sign — into +. In this way we will obtain the appropriate mathematical tools for the boson field.

4. — Permanents.

The counterpart of a determinant is a permanent. A « permanent » is quite similar to a determinant, the only difference being that each term of its development is taken with + sign. It is well known that the antisymmetrized product of N_0 wave functions $u^{(1)}, u^{(2)} \dots u^{(N_0)}$ is the determinant:

$$(9) \quad \begin{pmatrix} u^{(1)} & u^{(2)} & \dots & u^{(N_0)} \\ 1 & 2 & \dots & N_0 \end{pmatrix} = \begin{vmatrix} u^{(1)}(x_1) & u^{(1)}(x_2) & \dots & u^{(1)}(x_{N_0}) \\ u^{(2)}(x_1) & u^{(2)}(x_2) & \dots & u^{(2)}(x_{N_0}) \\ \dots & \dots & \dots & \dots \\ u^{(N_0)}(x_1) & u^{(N_0)}(x_2) & \dots & u^{(N_0)}(x_{N_0}) \end{vmatrix}$$

with $(u^{(h)}k) = u^{(h)}(x_k)$ and the symmetrized product of P_0 wave functions is the permanent:

$$(10) \quad \begin{bmatrix} z^{(1)} & z^{(2)} & \dots & z^{(p_0)} \\ 1 & 2 & \dots & p_0 \end{bmatrix} = \begin{vmatrix} z^{(1)}(x_1) & z^{(1)}(x_2) & \dots & z^{(1)}(x_{p_0}) \\ z^{(2)}(x_1) & z^{(2)}(x_2) & \dots & z^{(2)}(x_{p_0}) \\ \dots & \dots & \dots & \dots \\ z^{(p_0)}(x_1) & z^{(p_0)}(x_2) & \dots & z^{(p_0)}(x_{p_0}) \end{vmatrix}_+$$

with $[z^{(h)}k] = z^{(h)}(x_k)$.

5. - Hafnians.

In complete analogy, the counterpart of a pfaffian is a new algorithm that we will call a hafnian.

We define a hafnian with the formula

$$(11) \quad [1 \ 2 \dots 2n] = \sum' [i_1 i_2][i_3 i_4] \dots [i_{2n-1} i_{2n}].$$

Square brackets are used, consistently, for permanents and hafnians, round brackets for determinants and pfaffians.

For instance, for $n = 2$, eq. (11) gives:

$$[1 \ 2 \ 3 \ 4] = [1 \ 2][3 \ 4] + [1 \ 3][2 \ 4] + [1 \ 4][2 \ 3].$$

The triangular array is useful also for the hafnians:

$$[1 \ 2 \dots 2n] = \begin{vmatrix} [1 \ 2] & [1 \ 3] & \dots & [1 \ , \ 2n] \\ & [2 \ 3] & & [2 \ , \ 2n] \\ & & \ddots & \\ & & & [2n-1, \ 2n] \end{vmatrix}_+.$$

6. - Application to electrodynamics.

We wish to show how this formalism leads to a compact expression for the n -th order term of the perturbative expansion of Dyson's S -matrix.

In electrodynamics, with the usual meaning of the symbols, we have the expansion of Dyson's U matrix (in the interaction representation):

$$U(t_f, t_i) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_{t_i}^{t_f} d^4 x_1 \int_{t_i}^{t_f} d^4 x_2 \dots \int_{t_i}^{t_f} d^4 x_N \cdot T \left(\prod_{i=1}^N \psi_{\alpha_i}(x_i) \psi_{\beta_i}(x_i) A_{\mu_i}(x_i) \gamma'_{\alpha_i \beta_i} \right).$$

We want to find the transition amplitude between final and initial states $|f\rangle$ and $|i\rangle$.

Let us calculate first the vacuum-vacuum transition amplitude:

$$|f\rangle = |i\rangle = |0\rangle, \quad M_{00} = k_{00} = \langle 0 | U | 0 \rangle.$$

In the interaction representation the fields A and ψ commute. Therefore

we have to evaluate separately:

$$F = \langle 0 | T(\bar{\psi}_{\alpha_1}(x_1) \psi_{\beta_1}(x_1) \bar{\psi}_{\alpha_2}(x_2) \psi_{\beta_2}(x_2) \dots \bar{\psi}_{\alpha_N}(x_N) \psi_{\beta_N}(x_N)) | 0 \rangle .$$

$$B = \langle 0 | T(A_{\mu_1}(x_1) A_{\mu_2}(x_2) \dots A_{\mu_N}(x_N)) | 0 \rangle .$$

We start with the calculation of B . Call, for short, \prod the product $A_{\mu_1}(x_1) A_{\mu_2}(x_2) \dots A_{\mu_N}(x_N)$. Wick's theorem states that:

$$(12) \quad T(\prod) = : \prod : + \sum : \prod^{(2)} : + \sum : \prod^{(4)} : + \dots ,$$

where $\prod^{(h)}$ denotes any of the values \prod assumes when h pairs of operators are contracted, and $\sum \prod^{(2h)}$ denotes summation over all such values, for given h . It is then immediately seen that the only term of the expansion (12) which contributes to B is the term with all the A_{μ} are contracted, *i.e.*, with $h = n/2$. Therefore we see that only terms with even n will contribute to the perturbative expansion.

Then

$$(13) \quad B = \sum : \prod^{(n)} : .$$

By comparing eq. (11) with eq. (13) we get:

$$B = [1 \ 2 \ \dots \ 2n] ,$$

where $[hk] = \langle 0 | T(A_{\mu_h}(x_h) A_{\mu_k}(x_k)) | 0 \rangle = \frac{1}{2} \delta_{\mu_h \mu_k} D_F(x_h - x_k)$.

Denote now the product of the operators of the fermion field by:

$$p = 1 \ 2 \ 3 \ \dots \ 2n = \bar{\psi}_{\alpha_1}(x_1) \psi_{\beta_1}(x_1) \dots \bar{\psi}_{\alpha_n}(x_n) \psi_{\beta_n}(x_n) ,$$

where $h = 2(h) - 1$ stands for the operator $\psi_{\alpha_{(h)}}(x_{(h)})$, $h = 2(h)$ for $\bar{\psi}_{\beta_{(h)}}(x_{(h)})$ ($(h) = 1, 2, \dots, n$).

We denote with (α) the maximum integer contained in $(\alpha + 1)/2$.

In the same way as far the case of the boson field, we get: (remembering that $:\psi\psi: = :\bar{\psi}\bar{\psi}: = 0$)

$$(14) \quad F = (1 \ 2 \ \dots \ 2n) ,$$

where:

$$\begin{cases} (hk) = 0 & \text{for } h, k \text{ both even or both odd} \\ (hk) = \frac{1}{2} S_{\beta(h), \alpha(k)}^F(x_{(h)} - x_{(k)}) & \text{for } h \text{ odd, } k \text{ even} \\ (hk) = -\frac{1}{2} S_{\beta(h), \alpha(k)}^F(x_{(h)} - x_{(k)}) & \text{for } h \text{ even, } k \text{ odd} . \end{cases}$$

If we use the fundamental expansion theorem (7-bis) we can see that because of $(hk) = 0$ for h, k both even or both odd, the pfaffian reduces to a determinant.

Collecting all the results, one finds:

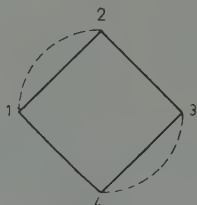
$$(15) \quad k_0 = M_{00} = \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} \int d1 \sum_{\alpha_1 \beta_1 \mu_1} \gamma_{\alpha_1 \beta_1}^{\mu_1} \dots \\ \dots \int d(2n) \sum_{\alpha_{2n} \beta_{2n} \mu_{2n}} \gamma_{\alpha_{2n} \beta_{2n}}^{\mu_{2n}} \begin{pmatrix} 1 & 2 & \dots & 2n \\ 1 & 2 & \dots & 2n \end{pmatrix} [1 \ 2 \dots 2n] .$$

This formula gives automatically the contribution of all Feynman diagrams without external lines.

For instance, in the fourth order, expanding the determinant and the hafnian one obtains, among many other terms,

$$\int d1 \sum_1 \gamma^1 \int d2 \sum_2 \gamma^2 \dots \int d4 \sum_4 \gamma^4 (1 \ 2)(2 \ 3)(3 \ 4)(4 \ 1) [1 \ 2] [3 \ 4]$$

which clearly gives the contribution of the graph:



A remarkable feature of expansion (15) is that the contribution of fermions is separated from the contribution of bosons.

7. - The general perturbative expansion.

So far we have obtained the perturbative expansion of k_0 , i.e., $\langle 0|U|0\rangle$. In essentially the same way, it is possible to obtain the element M_{FI} of the U matrix which describes the process in which

n electrons, n positrons, a photons

are destroyed and

p electrons, q positrons, b photons

are created, in states specified by some prescribed wave functions. Clearly $n+q=m+p$; we call N_0 this number, and set $P_0=a+b$.

We will give, without proof, a general formula which expresses M_{FI} in terms of the wave functions of the initial and final states (with the due properties of symmetry and normalization) and of a «kernel», whose form depends only upon N_0 and P_0 .

Let $u_\alpha(x)$ ($x=x_1, x_2, x_3, x_0$) be the wave function of an electron, $v_\beta(x)$ that of a positron, $\zeta_\mu(t)$ ($t=t_1, t_2, t_3, t_0$) that of a photon. (These functions being solutions of the free fields equations).

The adjoints of u and v are:

$$\bar{u} = u^* \gamma^4, \quad \bar{v} = \gamma^4 v^*.$$

The normalization of the spinor field is such that:

$$\int d^3x u^*(x) u(x) = 1, \quad \int d^3x v^*(x) v(x) = 1$$

and also the $\zeta(t)$ are normalized.

If now we want to describe an initial state with τ_1 photons in the state 1, τ_2 in the state 2 ..., τ_α in the state α , we can write the permanent (from now on, we shall use a short relation, neglecting all vector and spinor indices):

$$(16) \quad \varphi_I^{(b)} = \frac{1}{\sqrt{\tau_1! \tau_2! \dots \tau_\alpha! a!}} \left[\zeta^{(1)} \zeta^{(2)} \dots \zeta^{(a)} \right]_{t_1 \ t_2 \dots t_a} = \varphi(t_1 t_2 \dots t_a); \quad [\zeta^{(r)} t_h] = \zeta^{(r)}(t_h).$$

For final photons we have (σ_i photons in the state i' , etc.):

$$\varphi_F^{(b)} = \frac{1}{\sqrt{\tau_1'! \tau_2'! \dots \tau_{p'}! b!}} \left[\zeta^{(1')} \zeta^{(2')} \dots \zeta^{(b')} \right]_{t_{a+1} \ t_{a+2} \dots t_{p_0}} = \varphi(t_{a+1}, t_{a+2} \dots t_{p_0}).$$

Proceeding in the same way, for the initial and final electrons and positrons we get the determinants:

$$(18) \quad \varphi_I^{(b)}(y_1 y_2 \dots y_n) = \frac{1}{\sqrt{n!}} \begin{vmatrix} u^{(1)} & u^{(2)} & \dots & u^{(n)} \\ y_1 & y_2 & \dots & y_n \end{vmatrix}; \quad (u^{(k)} y_h) = u^{(k)}(y_h),$$

$$(19) \quad \varphi_I^{(b)}(x_1 x_2 \dots x_m) = \frac{1}{\sqrt{m!}} \begin{vmatrix} v^{(1)} & v^{(2)} & \dots & v^{(m)} \\ x_1 & x_2 & \dots & x_m \end{vmatrix}; \quad (v^{(k)} x_h) = v^{(k)}(x_h),$$

$$(20) \quad \varphi_F^{(b)}(x_{m+1} x_{m+2} \dots x_{N_0}) = \frac{1}{\sqrt{p!}} \begin{vmatrix} u^{(1')} & u^{(2')} & \dots & u^{(p')} \\ x_{m+1} & x_{m+2} & \dots & x_{N_0} \end{vmatrix},$$

$$(21) \quad \varphi_F^{(b)}(y_{n+1}, y_{n+2} \dots y_{N_0}) = \frac{1}{\sqrt{q!}} \begin{vmatrix} v^{(1')} & v^{(2')} & \dots & v^{(q')} \\ y_{n+1} & y_{n+2} & \dots & y_{N_0} \end{vmatrix}.$$

(The adjoints $\bar{\psi}$ are defined by writing in the Slater determinants the adjoints \bar{u} , \bar{v} of the single-particle wave functions u , v .)

We can finally give the result in the desired form. Define a kernel as the function:

$$(22) \quad K \left(\begin{matrix} x_1 x_2 \dots x_{N_0} \\ y_1 y_2 \dots y_{N_0} \end{matrix} \middle| \begin{matrix} t_1 t_2 \dots t_{P_0} \end{matrix} \right) = \\ = \sum_{N(P_0)} \frac{\lambda^N}{N!} \int d1 \sum_1 \gamma^1 \dots \int dN \sum \gamma^N \left(\begin{matrix} x_1 x_2 \dots x_{N_0} & 1 & 2 & \dots & N \\ y_1 y_2 \dots y_{N_0} & 1 & 2 & \dots & N \end{matrix} \right) [t_1 t_2 \dots t_{P_0} \quad 1 \quad 2 \dots N],$$

where $\sum_{N(P_0)}$ means summation over all the values of N having the same parity as P_0 .

Call Φ_I the product of (16), (18), and (19), Φ_F the product of (17), (20) and (21). Then we can write the matrix elements $M_{FI}(\tau_1 \tau_0)$ (τ_1 and τ_0 are times) as:

$$(23) \quad M_{FI}(\tau_1, \tau_0) = C_{FI} \hat{\int} \Phi_F K \left(\begin{matrix} x_1 x_2 \dots x_{N_0} \\ y_1 y_2 \dots y_{N_0} \end{matrix} \middle| \begin{matrix} t_1 t_2 \dots t_{P_0} \end{matrix} \right) \Phi_I,$$

where C_{FI} is simply a numerical coefficient and the integration is performed over all the variables $x_1 \dots x_{N_0}$, $y_1 \dots y_{N_0}$, $t_1 \dots t_{P_0}$. In the symbol $\hat{\int}$ is included a time average, which we introduce in order to avoid the adiabatic switching on and off of the charge. For instance if w is the wave function of an initial particle:

$$\hat{\int} \dots w(x) = \lim_{T \rightarrow \infty} \frac{1}{\tau_0 + T} \int_{-\tau}^{\tau_0} d\tau_0 \int d^3x \dots w(x).$$

Formula (22) gives all the contributions of all Feynman's graphs with N_0 external fermion lines and P_0 external boson lines.

8. -- Two particular cases.

We treat briefly the cases in which only the boson or only the fermion field is quantized.

1) *Boson field quantized.* In this case the bosons are in interaction with a classical current $j_\mu(x)$; the perturbation expansion can be summed exactly. No determinants appear in the expansion but only hafnians. For instance

the vacuum-vacuum probability amplitude is given by:

$$(24) \quad M_{00} = \sum_{n=0}^{\infty} \frac{(i)^{2n}}{(2n)!} \int d1 \sum \dots \int d(2n) \sum j(1)j(2) \dots j(2n) [1 \ 2 \dots 2n] = \\ = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n n!} \left[\int \int d1 \ d2 \ j(1) [1 \ 2] j(2) \right]^n = \exp \left[-\frac{1}{4} \int \int d1 \ d2 \ j(1) D^F(1-2) j(2) \right],$$

which is Glauber's result.

2) *Fermion field quantized.* In this case no hafnians appear in the expansion but only determinants. For instance the scattering of one electron by an external electromagnetic field $A(x)$ can be described by a kernel $\hat{K} \begin{pmatrix} x \\ y \end{pmatrix}$ whose perturbative expansion is:

$$(25) \quad \tilde{K} \begin{pmatrix} x \\ y \end{pmatrix} = \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \int d1 \sum_1 \dots \int dN \sum_N \gamma^\mu A_\mu(1) \dots \gamma^\mu A_\mu(N) \begin{pmatrix} 1 \ 2 \dots Nx \\ 1 \ 2 \dots Ny \end{pmatrix}.$$

This is just the Fredholm expansion and it is well known that:

$$(26) \quad \hat{K} \begin{pmatrix} x \\ y \end{pmatrix} = (xy) \tilde{K}_0 - \lambda \int (x1) \gamma^\mu A_\mu(1) \tilde{K} \begin{pmatrix} 1 \\ y \end{pmatrix},$$

where

$$K_0 = \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \int d1 \sum_1 \dots \int dN \sum_N \gamma^\mu A_\mu(1) \dots \gamma^\mu A_\mu(N) \begin{pmatrix} 1 \ 2 \dots N \\ 1 \ 2 \dots N \end{pmatrix}.$$

(The usual notations are $\tilde{K} \begin{pmatrix} x \\ y \end{pmatrix} / \tilde{K}_0 = K^A(xy)$).

9. - Branching equations.

Eq. (26) can be deduced from the expansion (25), and eq. (26) is just Fredholm's equation. We want now to deduce from the general expansion (22) all possible equations among kernels corresponding to different values of N_0 and P_0 .

These equations can immediately be obtained by expanding the determinants by rows and columns, and the hafnians by lines.

We give here only two examples of these equations (it would be easy to write down all possible independent equations):

$$(27) \quad K \begin{pmatrix} x \\ y \end{pmatrix} = K_0(xy) - \lambda \int \sum_1 d1 \gamma^1(x1) K \begin{pmatrix} 1 \\ y \ 1 \end{pmatrix},$$

$$(28) \quad K \begin{pmatrix} x \\ y \end{pmatrix} \Big|_{t_1 t_2} = [t_1 t_2] K \begin{pmatrix} x \\ y \end{pmatrix} + \lambda \int \sum_1 d1 \gamma^1[t_1 1] K \begin{pmatrix} x \ 1 \\ y \ 1 \ t_2 \end{pmatrix}.$$

We see that eq. (27) is similar but not identical with eq. (26). And in fact this theory is more general than Fredholm's theory.

There is also another set of equations; which we can derive from eq. (22), *i.e.*,

$$(29) \quad \frac{d}{d\lambda} K \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| t_1 \dots t_{P_0} \right) = \int \sum_1 d1 \, \gamma^1 K \left(\begin{matrix} x_1 \dots x_{N_0} 1 \\ y_1 \dots y_{N_0} 1 \end{matrix} \middle| 1 t_1 \dots t_{P_0} \right).$$

Instead of (29), in the Fredholm theory we had:

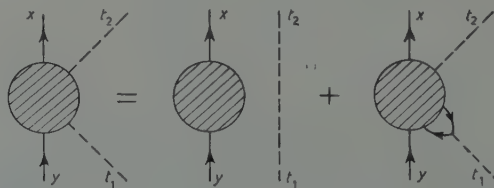
$$(30) \quad \frac{d}{d\lambda} K \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \right) = \int d1 \, K \left(\begin{matrix} x_1 \dots x_{N_0} 1 \\ y_1 \dots y_{N_0} 1 \end{matrix} \right).$$

Formula (29) plays an important role in the study of renormalization.

We want now to show a simple total graph interpretation of the branching equations: for the eq. (27) we get the graph:



The total graph interpretation of eq. (28) is:



Finally, let us say a few words about the renormalization and the convergence of the perturbative expansion.

It is possible to show that if the concept of integral is suitably redefined (in a way which is analogous to Hadamard's *partie finie*) then it follows automatically that the formal, unrenormalized theory, becomes *ipso facto* the renormalized theory. Any such prescription—infinite choices are possible—can be proved to be equivalent, (to within additional finite renormalizations), to the formal use of counter terms.

Of course, nobody knows whether the renormalized expansion converges or not. It is interesting to notice, however, that if we take a non-singular

theory (4-integral volume Ω finite); $|(xy)| < M$; $[[xy]] < L$ the perturbative expansion has a finite radius of convergence. If, further the number of free modes of the field is finite, then it can be proved that the radius of convergence of (22) is infinite, and everything works as in the Fredholm theory.

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Vacuum Expectation Values and Analytic Functions.

Notes by

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1. - Definition of a field.

(For a more complete discussion see A. S. WIGHTMAN: *Problèmes Mathématiques de la théorie quantique des champs*. Hectographed notes on lectures obtainable from Ecole Normale Supérieure, Laboratoire de Physique, 1957).

We first discuss the characterization and properties of field operators $A(x)$. From a physical point of view the really significant quantities are $A(f) = \int A(x) f(x) dx$ where the test functions $f(x)$ are smooth and vanish outside of some region. From the example of the scalar field we see that the only reasonable physical assumption about $A(f)$ is that it must be, in general, an unbounded operator. There are some vectors for which $(A(f)\varphi, A(f)\varphi) = \infty$.

A common domain D of all $A(f)$ ought to exist satisfying:

- 1) Ψ_0 (physical vacuum) $\in D$.
- 2) $A(f)D \subset D$, i.e., $A(f)$ applied to a vector in D yields a vector in D .
- 3) D must be Lorentz-invariant,

$$U(a, \Lambda)D \subset D.$$

The field itself has the transformation law

$$U(a, \Lambda)A(f)U(a, \Lambda)^{-1} = A(f_{\{a, \Lambda\}}),$$

with the following transformation property of the test functions

$$f_{\{a, A\}}(x) = f(A^{-1}(x - a)) .$$

4) D is linear.

5) D is dense; i.e. vectors in D get arbitrarily close to any vector.

Thus we can take $A(f)$ to belong to $L(D, D)$, the space of linear mappings of D into D , and give the following definition of a field: « A scalar field is a distribution on space-time with values in $L(D, D)$ ».

To complete this definition, we must introduce a notion of continuity. For measurable quantities we require that if a sequence $f_n \rightarrow 0$, then $\lim_{n \rightarrow \infty} (\Psi, A(f_n)\Psi) = 0$. This continuity implies also that of $(\Psi, A(f)\Phi)$, (weak continuity); consequently one can introduce a topology in $L(D, D)$.

Vector or spinor fields. — In this case we put, for example

$$\int \sum_{\mu=0}^3 f^{\mu}(x) v_{\mu}(x) dx = v(f)$$

or if T is a distribution with the transformation property

$$U(a, A) T_{\alpha}(x) U(a, A)^{-1} = \sum_{\beta=1}^{\dim S(\alpha)} S(A^{-1})_{\alpha\beta} T_{\beta}(Ax + a)$$

we set

$$\int \sum_{\alpha} f_{\alpha}(x) T_{\alpha}(x) dx = T(f) .$$

The transformation property of the test functions is now

$$f_{\{a, A\}} = \sum_{\beta=1}^{\dim S} S(A^{-1})_{\alpha\beta}^T f_{\beta}(A^{-1}(x - a)) .$$

Then the general formula

$$U(a, A) T(f) U(a, A)^{-1} = T(f_{\{a, A\}})$$

is again valid.

2. — Vacuum expectation values.

From the above assumptions it follows that the following multilinear functionals exist: (symbols of summation over repeated indices are hereafter omitted)

$$(\Psi_0, T_1(f_1) \dots T_n(f_n)\Psi_0) = \int dx_1 \dots dx_n f_{1x_1}(x_1) \dots f_{nx_n}(x_n) (\Psi_0, T_{1x_1}(x_1) \dots T_{nx_n}(x_n)\Psi_0) .$$

By Schwartz's nuclear theorem this formula has an extension to test functions of the form $f_{\alpha_1 \dots \alpha_n}(x_1 \dots x_n)$. We also introduce the distributions:

$$F_{\alpha_1 \dots \alpha_n}^{T_1 \dots T_n}(x_1 \dots x_n) = (\Psi_0, T_{1\alpha_1}(x_1) \dots T_{n\alpha_n}(x_n) \Psi_0) = F_{\alpha_1 \dots \alpha_n}^{T_1 \dots T_n}(x_1 - x_2, \dots, x_{n-1} - x_n),$$

where the last equality follows from the translational invariance as can easily be shown.

It can be shown also that F is bounded in each variable separately. No proof as yet has been given that boundedness in all variables together follows from the above assumptions. But we shall make the assumption (or conjecture) that $F_{\alpha_1 \dots \alpha_n}^{T_1 \dots T_n}$ can be extended to $\mathcal{S}_{\xi_1 \dots \xi_{n-1}}^*$.

Properties of the Fourier transform

$$F_{\alpha_1 \dots \alpha_n}^{T_1 \dots T_n}(\xi_1 \dots \xi_{n-1}) = \int \exp \left[-i \sum_{j=1}^{n-1} p_j \xi_j \right] G_{\alpha_1 \dots \alpha_n}^{T_1 \dots T_n}(p_1 \dots p_{n-1}) \quad \xi_j = x_j - x_{j+1}.$$

If each p_j is not in the physical spectrum (see WIGHTMAN - Pages 91-92) then $G = 0$.

3. - Analytic continuation.

We replace the variables ξ_j by $\xi_j - i\eta_j$. This is then the Laplace transformation of the distributions

$$F_{\alpha_1 \dots \alpha_n}^{T_1 \dots T_n}(\xi_1 - i\eta_1, \dots, \xi_{n-1} - i\eta_{n-1}) = \int \exp \left[-i \sum_{j=1}^{n-1} p_j (\xi_j - i\eta_j) \right] G_{\alpha_1 \dots \alpha_n}^{T_1 \dots T_n}(p_1 \dots p_{n-1}),$$

where η_j lies inside the light cone (see LIONS, - Pages 54-58).

F as a function of $z_j = \xi_j - i\eta_j$ is then an analytic function in the « tube » $\mathcal{T}_{n-1}(z_1 \dots z_{n-1})$ with η_j in future cone.

Under the orthochronous Lorentz transformations, $A \in L_+^\uparrow$

$$\sum_{\beta_1 \dots \beta_n} \prod_{j=1}^n S_{\alpha\beta}^{(j)}(A) F_{\beta_1 \dots \beta_n}^{T_1 \dots T_n}(z_1 \dots z_{n-1}) = F_{\alpha_1 \dots \alpha_n}^{T_1 \dots T_n}(Az_1, \dots, Az_{n-1}).$$

We can now continue A into the complex domain too: A will again satisfy $AGA^T = G$, but will have complex elements. We denote this group by « complex Lorentz group ». Then the quantities (Az_1, \dots, Az_{n-1}) define a larger manifold which we call the « extended tube » \mathcal{T}'_{n-1} . We note also that the complex Lorentz group has only two pieces in contrast to the real Lorentz group which consists of four pieces, a fact which is connected with the CPT theorem.

The following question arises: can we maintain the above transformation law if $F(Az)$ and $S(A)$ are continued analytically separately? Using the well-

known theorem that if two analytic functions agree on a real domain, and have a complex neighborhood of that domain common to their domain of definition, they coincide, we can answer the question positively at least in the neighborhood of $A = I$. One can also show that the above transformation property holds everywhere on the complex Lorentz group. Thus we have the result: « F is analytic and single-valued in the extended tube \mathfrak{T}'_{n-1} ». (*)

An analytic function invariant under the orthochronous real Lorentz group $F^{T_1 \dots T_n}$ is actually an analytic function of the Lorentz scalar products $z_j \cdot z_k$. From this fact it is then a trivial matter to see that F is analytic in the extended tube \mathfrak{T}'_{n-1} . However, not all the scalar products are independent, and, for large n , it may be advantageous to work with the variables z_1, \dots, z_{n-1} rather than with the scalar products.

There are no real points in the tube \mathfrak{T}_n . However, the extended tube \mathfrak{T}'_n has real points (« Jost-points ») given by the following theorem.

Theorem (JOST: *Helv. Phys. Acta*, **30**, 409 (1957)). — « z_1, \dots, z_n is a real point of \mathfrak{T}'_n if and only if all vectors $\sum_{i=1}^n \lambda_i z_i$, with $\lambda_i \geq 0$, $\sum \lambda_i = 1$ are space-like ».

$(\Psi_0, T_1(x_1) \dots T_n(x_n) \Psi_0)$ is analytic at Jost-points in the sense that it has a convergent power-series expansion. And the knowledge of a vacuum expectation value in the neighborhood of a Jost-point determines it everywhere.

4. — Local commutativity.

We consider for simplicity scalar fields,

$$[T_j(x), T_k(y)] = 0; \quad (x - y)^2 < 0.$$

This implies that

$$F^{T_1 \dots T_j T_{j+1} \dots T_n}(\xi_1, \dots, \xi_{n-1}) = F^{T_1 \dots T_{j+1} T_j \dots T_n}(\xi_1, \dots, \xi_{j-1} + \xi_j, -\xi_j, \xi_j + \xi_{j+1}, \dots).$$

The function on the left hand side is analytic in the region \mathfrak{T}'_{n-1} , while that on the right side is analytic in the region $P(j, j+1)\mathfrak{T}'_{n-1}$ obtained from \mathfrak{T}'_{n-1} by the transformation:

$$z_{j-1} \rightarrow z_j + z_{j-1}, \quad z_j \rightarrow -z_j, \quad z_{j+1} \rightarrow z_j + z_{j+1}, \quad z_k \rightarrow z_k, \quad k \neq j, j-1, j+1.$$

Using this equality we can extend F from \mathfrak{T}'_{n-1} to $P_{j,j+1}\mathfrak{T}'_{n-1}$ and continuing

(*) See D. HALL and A. S. WIGHTMAN: *Dan. Mat. Fys. Medd.*, **31**, No. 5 (1957).

this process we obtain the following result:

$$\ll F^{T_1 \dots T_n} \text{ is analytic in } \bigcup_{g \in S_n} P(g) \mathcal{C}'_n \gg$$

where S_n is the permutation group of n objects ».

We can replace the requirement of local commutativity by that of the analyticity in a certain domain. For the three-fold expectation values it turns out that the local commutativity for all space-like separations follows from local commutativity in a neighborhood of any Jost point. Whether this is true for higher order vacuum expectation values is, at present, not known (*). However, we have the following

Theorem (DYSON). — « If weak local commutativity (WLC) holds for all space-like separations then the vacuum expectation values are analytic for all such points and vice versa ».

Note: WLC means

$$F^{T_1 \dots T_n}(\xi_1, \dots, \xi_{n-1}) = F^{T_n \dots T_1}(-\xi_{n-1}, \dots, -\xi_1).$$

5. — Holomorphy domains. Holomorphy envelopes.

It is a well-known result in the theory of analytic functions (see for example BIBERBACH: *Lerhbuch der Funktionentheorie*, II) that given any domain one can find a function $f(z)$ such that it is analytic in the given domain and has singularities on the boundary. On the other hand, for functions of several complex variables there are domains such that *any* function analytic inside is also analytic in a large domain. Consider a class of functions analytic in a domain, then there exists a largest domain to which all these functions can be continued analytically. This largest domain is called the « holomorphy envelope » of the original domain.

For a discussion of this idea see BEHNKE THULLEN: *Ergeb. der Math.* (1931).

The holomorphy envelope for the domain $\bigcup_{g \in S_3} \mathcal{D}(g) \mathcal{C}'_2$ is computed in G. KÄLLÉN and A. S. WIGHTMAN: *Dan. Mat. Fys. Skr.*, 1, n. 6 (1958).

(*) *Note added in proof.* — It was stated as true in the speaker's contribution to Colloque sur les Problèmes Mathématiques de la Théorie Quantique des Champs. (Lille, 1957) but the argument given was insufficient. O. STEINMANN and R. JOST have recently proved it (private communication).

Covariant Formalism of a Field with Indefinite Metric.

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The aim of this seminar is to try to investigate in a concrete way what is the structure of a relativistic field which contains «ghost» states; that is to say states with zero norm.

As we do not know what happens for the case of interacting fields, we shall restrict ourselves in the following to the case of free fields, which can be thought of as asymptotic parts of a more complicated theory.

1. Derivation of a model of «dipole-ghost» field.

We have heard from Heisenberg's lectures what a dipole ghost is, and let us recall its fundamental properties.

There are two states A and B which satisfy the metric relations:

$$(1) \quad (A, B) = 1, \quad (A, A) = (B, B) = 0$$

and satisfy also, H being the Hamiltonian operator,

$$(2) \quad HA = EA, \quad HB = EB + CB \quad E, C \text{ constants.}$$

To get a similar result derived from a canonical formalism, let us start from the Lagrangian density of a classical field:

$$(3) \quad \mathcal{L}(x) = \frac{1}{2i} (\partial_\mu \varphi \partial^\mu \varphi - \mu^2 \varphi^2) - \frac{1}{2i} (\partial_\mu \varphi^* \partial^\mu \varphi^* - \mu^{*2} \varphi^{*2}),$$

which describes two particles with complex conjugate masses. Now, make

(*) Now at CEN Saclay (France).

the transformation:

$$(4) \quad \varphi = \frac{A + iB}{\sqrt{2}}, \quad \varphi^* = \frac{A - iB}{\sqrt{2}},$$

$$(5) \quad \mathcal{L}(x) = \partial_\mu A \partial^\mu B - \operatorname{Re} \mu^2 AB - \operatorname{Im} \mu^2 \left(\frac{A^2 - B^2}{2} \right).$$

If now we want to go to the case of μ^2 real, in order to get the « dipole », it is convenient to do the transformation

$$A \rightarrow \alpha A, \quad B \rightarrow \frac{B}{\alpha},$$

so that

$$(6) \quad \mathcal{L}(x) = \partial_\mu A \partial^\mu B - \operatorname{Re} \mu^2 AB - \frac{\operatorname{Im} \mu^2}{2} \left(\alpha^2 A^2 - \frac{B^2}{\alpha^2} \right).$$

Now let $\operatorname{Im} \mu^2 \rightarrow 0$ in such a way that $\operatorname{Im} \mu^2 \alpha^2 \rightarrow \lambda^2$ and $\lim \operatorname{Re} \mu^2 = m^2$. We get finally the Lagrangian which was found first by V. GLASER

$$(7) \quad \mathcal{L}(x) = \partial_\mu A(x) \partial^\mu B(x) - m^2 A(x) B(x) - \frac{\lambda^2}{2} A^2(x).$$

Let us now quantize this field in the usual canonical way. The equations of motion are

$$(8) \quad (\square + m^2)A(x) = 0, \quad (\square + m^2)B(x) = -\lambda^2 A(x).$$

The commutation relations are derived in the usual way

$$(9) \quad [A(x), A(x')] = [\dot{A}(x), A(x')] = 0, \quad \text{for } t = t'$$

so by virtue of (8)

$$(10) \quad [A(x), A(x')] = 0,$$

$$(11) \quad [A(x), B(x')] = 0 \quad \text{and} \quad [\dot{A}(x), B(x')] = -i \delta(x - x') \quad \text{for } t = t'$$

so that by virtue again of (8):

$$(12) \quad [A(x), B(x')] = i \Delta(x - x'; m^2).$$

And we have now the last commutation relation:

$$(13) \quad [B(x), B(x')] = [\dot{B}(x), B(x')] = 0 \quad \text{for } t = t'.$$

And it is easy to verify that by (8) and (12)

$$(14) \quad [B(x), B(x')] = i\lambda^2 \frac{\partial}{\partial m^2} \Delta(x-x'; m^2),$$

as we can infer from

$$(\square + m^2)\Delta(x) = 0 \quad \text{that} \quad (\square + m^2) \left[\frac{\partial}{\partial m^2} \Delta(x) \right] = -\Delta(x).$$

Now we can ask for the momentum representation of these fields. Let us develop, in agreement with (8):

$$(15) \quad A(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int A(k) \delta(k^2 - m^2) \exp[-ikx] d_4 k.$$

Eq. (10) gives obviously:

$$(16) \quad [A(k), A(k')] = 0.$$

Now let us write B in the following way:

$$(17) \quad B(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int B(k) \delta(k^2 - m^2) \exp[-ikx] d_4 k + B'(x),$$

and put in accordance with (1):

$$(18) \quad [A(k), B(k')] \delta(k^2 - m^2) = \varepsilon(k_0) \delta(k + k').$$

Then it follows from (12) that $[A(x), B'(x')] = 0$.

So we can express $B'(x)$ in terms of $A(k)$.

And it is easy to check that the convenient expression which leads to (14) is:

$$B'(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{\lambda^2}{m^2} \int A(k) \binom{2 - ikx}{2} \exp[-ikx] \delta(k^2 - m^2) d_4 k$$

if we put in accordance with (1)

$$(19) \quad [B(k), B(k')] = 0.$$

So that finally:

$$(20) \quad B(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \left[B(k) + \frac{\lambda^2}{m^2} \binom{2 - ikx}{2} A(k) \right] \delta(k^2 - m^2) \exp[-ikx] d_4 k.$$

Let us now look for the energy-momentum operator:

$$\begin{aligned}\text{We define it by} \quad [P_\mu, A(x)] &= -i \partial_\mu A(x), \\ [P_\mu, B(x)] &= -i \partial_\mu B(x).\end{aligned}$$

And it is most conveniently written in the momentum representation as:

$$(21) \quad P_\mu = \int \varepsilon(k_0) k_\mu \left[A(k) B(-k) + \frac{\lambda}{4m^2} A(k) A(-k) \right] \delta(k^2 - m^2) d_4 k.$$

The nature of the first term is due to the particular nature of the commutation relation (18). The appearance of the second term is due to the linear term in x in the development (20), and gives rise to the particular commutation relations

$$(22) \quad [P_\mu, A(k)] = -k_\mu A(k); \quad [P_\mu, B(k)] = -k_\mu \left[B(k) + \frac{\lambda^2}{2m^2} A(k) \right],$$

which strongly remind us of relations (2).

These last relations (22) allow us to define a vacuum: it must be the lowest eigenvector of P_0 . So that certainly:

$$(23) \quad \theta(k_0) A(k) |0\rangle = 0$$

because otherwise the application of $A(k)$ ($k_0 > 0$) would transform it to a lower eigenvector. But the condition is less evident for the $B(k)$ operator.

Anyway let us assume that for some given q , one has: $B(q)|0\rangle = 0$. Then certainly: $[P_0, B(q)]|0\rangle = 0$ so that $(\lambda^2/m^2) A(q)|0\rangle = 0$ by virtue of (22).

And therefore $q_0 > 0$ because of (23).

So the only coherent condition we can add to (23) to define the vacuum is:

$$(24) \quad \theta(q_0) B(q) |0\rangle = 0.$$

Now we can pass to the usual formalism of creation and annihilation operators and write:

$$(25) \quad \begin{cases} A(k) = a(\mathbf{k})\sqrt{2\omega}, & B(k) = b(\mathbf{k})\sqrt{2\omega} & \text{for } k_0 = \omega > 0, \\ A(k) = a^*(-\mathbf{k})\sqrt{2\omega}, & B(k) = b^*(-\mathbf{k})\sqrt{2\omega} & \text{for } k_0 = -\omega < 0. \end{cases}$$

This defines only a linear structure of the state vector space. We have to define a metric on it.

To do it, it suffices to define a bra-vacuum $\langle 0|$ by the relations

$\langle 0 | a^*(\mathbf{k}) = 0$, $\langle 0 | b^*(\mathbf{k}) = 0$ and $\langle 0 | 0 \rangle = 1$. We say then that the conjugate of any state $a^*(k_1) \dots b^*(q_1) \dots | 0 \rangle$ is $\langle 0 | \dots b(q_1) \dots a(k_1)$. And we compute the scalar products of two vectors by pushing the creation operators to the left, and the annihilation operators to the right, as usual. For example, the « squared norm » of $a^*(\mathbf{k}) | 0 \rangle$ is $\langle 0 | a(\mathbf{k}) a^*(\mathbf{k}) | 0 \rangle = 0$ and the same for $b^*(\mathbf{k}) | 0 \rangle$, but these two vectors are not orthogonal, as $\langle 0 | a(\mathbf{k}) b^*(\mathbf{k}') | 0 \rangle = \delta(\mathbf{k} - \mathbf{k}')$.

We define also the pseudo-hermitian conjugate H^* of an operator H by

$$(\varphi, H\psi) = (\psi, H^*\varphi)^*.$$

2. Study of a scattering matrix. Exclusion of the ghosts.

Let us now assume that the field we have just described is the asymptotic part of some theory which contains also other physical particles, which correspond to the usual metric. Then the total Hilbert space is the direct product of the Hilbert space of the ghosts which we have just defined, and of that one of the physical particles.

But if we have an S -matrix, in terms of these asymptotic fields, it will not be in general reducible with respect to this direct product; that is to say, it will in general transform a physical state in a mixture of physical and ghost states. And that is a great trouble because if we compute the transition probability from a state φ to a state ψ , it is:

$$\frac{|\langle \varphi | S | \psi \rangle|^2}{\langle \varphi | \varphi \rangle \langle \psi | \psi \rangle}$$

and if ψ is a ghost, it may happen that $\langle \psi | \psi \rangle < 0$ and it is physically not understandable to have a negative probability.

To avoid this unphysical result, we shall try the following procedure, which is essentially what is done by Heisenberg in his treatment of the Lee-model: we can avoid any unphysical result if we can restrict the space of final states to a space with positive (semidefinite) metric. But to compensate these restrictions, we can allow the initial state to be in a space larger than the physical vector state space, with the condition that the metric in that space be also positive.

So we are led to the following problem:

Can one find a subspace of the total Hilbert space which satisfies the conditions:

- a) it has a positive (perhaps semi-definite) metric;
- b) to each physical state, one can associate a vector of it, which con-

tains the physical state vector, and has the same norm, and which is transformed by the S -matrix in a vector of the same space.

Such a Hilbert space we call H_I , the rest of the initial Hilbert space is called H_{II} .

We can do this separation only in the Hilbert space of ghost states, and then do the direct product with the Hilbert space of physical particles.

Let us now consider two necessary conditions on H_I :

1) All vectors of H_I except the vacuum, which represents the physical particles, have zero norm. They are all orthogonal to each other.

2) The energy-momentum operator must transform H_I into itself, so that the definition of H_I is invariant under translations.

Let us study these two conditions in the Fock representation.

In the case of one ghost, the most general state vector is:

$$\int \varphi(\mathbf{k}) a^*(\mathbf{k}) + \psi(\mathbf{k}) b^*(\mathbf{k}) |0\rangle d\mathbf{k}.$$

The norm of it is:

$$\int [\varphi^*(\mathbf{k})\psi(\mathbf{k}) + \psi^*(\mathbf{k})\varphi(\mathbf{k})] d\mathbf{k},$$

so that to be sure that it has zero norm and that it is orthogonal to all others we have to take either all $\psi(k) = 0$ or all $\varphi(k) = 0$. This second condition is not invariant by translation, because

$$P_\mu \int \psi(\mathbf{k}) b^*(\mathbf{k}) |0\rangle d\mathbf{k} = \int k_\mu \psi(\mathbf{k}) \left[b^*(\mathbf{k}) + \frac{\lambda^2}{2m^2} a^*(\mathbf{k}) \right] d\mathbf{k} |0\rangle.$$

So the only choice in this sector of the Fock representation is:

$$\int \varphi(\mathbf{k}) a^*(\mathbf{k}) d\mathbf{k} |0\rangle \in H_I.$$

Likewise, if we have 3 ghosts, we have 4 independent amplitudes, abbreviated as a^{*3} , $a^{*2}b^*$, a^*b^{*2} and b^{*3} .

Of these a^{*3} and b^{*3} are not orthogonal and $a^{*2}b^*$, a^*b^{*2} neither. But P_μ transforms a^{*3} into itself, $a^{*2}b^*$ into a mixture of $a^{*2}b^*$ and a^{*3} and so on. So the only choice we can make is a^{*3} and $a^{*2}b \in H_I$. For the odd numbers of ghosts, the separation between H_I and H_{II} is unique: all states which contain more a^* operators than b^* operators belong to H_I .

For an even number, it is almost the same rule, with only a slight com-

plication for the case of equal numbers of a^* and b^* . We have to adopt a criterion in that subspace which tells us how to do the separation, and this criterion is in a very large measure arbitrary. We shall not discuss it here, as it does not change sensibly the main lines of the argument.

Now, how can we derive from the formal, pseudo unitary S -matrix as physical one, which is unitary?

Let us start from an initial physical state $|i\rangle$ and add to it an arbitrary, yet undetermined $|z_i\rangle$ vector of H_I with zero norm. All physical results are left unchanged by this addition. Apply then to $|i\rangle + |z_i\rangle$ the S -matrix, we get in general the sum of three vectors: $|f\rangle$ which is physical $|z_f\rangle$ which has zero norm and belongs to H_I and $|II_f\rangle$ which belongs to H_{II} .

Now the problem is to determine $|z_i\rangle$ in such a way that $|II_f\rangle$ vanishes, so that $|f\rangle$ then represents a physical state, which has the same properties of norm and energy as $|f\rangle + |z_f\rangle$. That we can in general do, because of a formal conservation of energy momentum: it will perhaps appear in the ghost matrix elements of the S -matrix some derivations of the δ -functions of the total momentum, due to the particular nature of the dipole-ghost, but for any finite energy process, a finite number of ghosts can enter, so that the condition that $|II_f\rangle = 0$ is expressed as a finite number of equations on the initial arbitrary vector $|z_i\rangle$. And one can convince oneself that this finite number of equations is the same as the number of arbitrary functions we have in $|z_f\rangle$ because for each number of particles, there are as many amplitudes in H_I as in H_{II} . So that one can hope that there is a unique result.

This procedure determines therefore $|f\rangle$ in terms of $|i\rangle$. And as these two vectors are physical, and as the norm and the energy are equal, there is a matrix \tilde{S} which is unitary, so that $\tilde{S}|i\rangle = |f\rangle$ for all i . This matrix has the properties of a physical S -matrix.

3. - Case of complex masses.

The Lagrangian (3) gives rise, following the canonical procedure, to a different kind of ghost field, namely to a field where all states have complex energies. The spectrum in energy is symmetrical with respect to the real axis, and each state with complex energy has zero norm, but is not orthogonal to that with conjugate complex energy. In terms of commutation relations it gives,

$$[P_\mu, a^*(\mathbf{k})] = k_\mu a^*(\mathbf{k}), \quad [P_\mu, b^*(\mathbf{k})] = k_\mu^* b^*(\mathbf{k}),$$

$$[a(\mathbf{k}), a'(\mathbf{k}')] = 0, \quad [b(\mathbf{k}), b^*(\mathbf{k}')] = 0, \quad [a(\mathbf{k}), b^*(\mathbf{k}')] = \delta(\mathbf{k} - \mathbf{k}'),$$

where a^* , b^* are some convenient creation operators, and

$$k_i = \mathbf{k}_i, \quad k_0 = \sqrt{\mathbf{k}^2 + \mu^2}, \quad k_0^* = \sqrt{\mathbf{k}^2 + \mu^{*2}}.$$

Of course it is impossible in such a formalism to define, even formally, an S -matrix, because of the appearance of divergent real exponentials in time.

A method in order to get a physical S -matrix is the following (it is due to G. KÄLLÉN and W. PAULI): let us consider the U -matrix referring to a finite interval of time $U(T, -T)$. It is, at least formally, well defined, as in any other field theory, because the presence of exponentials here does not lead to any trouble, T being finite.

Then we define as Hilbert space I all those states which have their energy in the upper half-plane, for example, and Hilbert space II the states which have it in the lower halfplane.

Now we can, at least formally, write down the same equations which we have written above, that is:

$$|II_f\rangle = 0.$$

This procedure works perfectly well in the non-relativistic case, where we have a finite number of ghost states. But here the trouble is that in a relativistic theory, there are always an infinity of ghost states possible, as we cannot use for a finite time the conservation of energy, as we did for the dipole case in order to limit the number of amplitudes which one has to take into account.

4. — Conclusions.

We have seen in this study that the idea of ghosts can be associated in a coherent way to a covariant formalism of free fields. As we have interpreted them as in- or outgoing fields, we have seen that it is possible to eliminate them of the physical results, but it seems that, if one has to do explicit calculations in any more concrete theory, it would be much easier to handle the dipole ghosts than the exponential ghosts.

Report and Comment on F. Gürsey's « Group Structure of Elementary Particles ».

W. PAULI [†] and B. TOUSCHEK

The contribution of W. PAULI to this report was not intended for publication. However, it was decided to publish it, in the form in which the talk was given, as a document of his last activities.

1. - Introduction (PAULI).

The fields of all the particles are to be constructed from the minimum number of fields. We want the answers to the questions: how do spin and iso-spin interplay, and is it reasonable to extrapolate iso-spin to the leptons? Is it possible that the electron mass is purely electromagnetic, with no contribution from other interactions? If this is the case, γ_5 invariance would mean that the electromagnetic mass is also zero; this raises doubts as to the possibility of this.

Scheme A. - The basic fields are 3 four-component spinors, two leptons and one baryon. This can account for the continuous group, but fails to incorporate C and P (which are discontinuous groups) for strong interactions. Schemes B and C have 4 four-component spinors, but these give too many particles.

Scheme D. - Mixed model.

- λ 2-component (iso-singlet) spinor,
- Λ four-component (iso-singlet) spinor,
- $k_1 k_2$ isodoublet scalar meson.

The elementary particles are compound states of these fields. Perhaps this scheme is less convincing, as it does not relate invariance under isotopic rotations to the invariance of the fields, and because the connection with my transformations for Dirac-fields with zero mass is here lost.

This treatment is group-theoretical and says nothing about the interactions.

Scheme A. — Let $L = \frac{1}{2}(1 + \gamma_5)$, $R = \frac{1}{2}(1 - \gamma_5)$ and let ε , V be lepton fields and Λ the baryon field. Write ε_L for $L\varepsilon$.

Introduce the 2×2 matrices F and G

$$F = (V_L, \varepsilon_L), \quad G = (-i\sigma_2 \varepsilon_R^*, i\sigma_2 V_R^*)$$

and the notation

$$\bar{F} = \sigma_2 F \sigma_2.$$

Form now the Lorentz invariant matrix $\theta = \bar{F}G$.

Two Lorentz-invariant rotation groups may now be defined

$$(I) \quad F \rightarrow FR, \quad G \rightarrow G,$$

$$(II) \quad F \rightarrow F, \quad G \rightarrow GQ,$$

where R and Q are unimodular and unitary: $R = \exp[i\sigma \cdot t]$, $Q = \exp[i\sigma \cdot u]$.

However, GÜRSEY does not show that this exhausts the possibilities. Combining the two 3-dimensional rotations gives a 4-dimensional rotation

$$\theta \rightarrow \bar{R}\theta Q, \quad \bar{\theta} \rightarrow \bar{Q}\bar{\theta}R, \quad \theta^+ \rightarrow Q\theta^+R,$$

and $\text{Tr}(\theta\bar{\theta})$ is invariant.

The elements of θ represent four spin-zero boson fields

$$\begin{aligned} \theta^0 &= \varepsilon_R^+ \varepsilon_L, \quad \theta^+ = \varepsilon_R^+ \nu_L, & \theta &= \begin{pmatrix} \theta^0 & -\theta^+ \\ -\theta^+ & \bar{\theta}^0 \end{pmatrix}, & \bar{\theta} &= \begin{pmatrix} \bar{\theta}^0 & \theta^- \\ \theta^- & \theta^0 \end{pmatrix}. \\ \theta^- &= \nu_R^+ \varepsilon_L, \quad \theta^0 = \nu_R^+ \nu_L, \end{aligned}$$

One can identify (I) with the iso-spin group, and θ^+ , θ^0 behave as isospinor bosons. GÜRSEY liked (II) also, as it agrees with the 4-dimensional idea of Gell-Mann and Schwinger.

Iso-vector Boson. — This is constructed out of the matrix θ

$$\Phi \cdot \sigma = \frac{1}{2}(\theta\theta^+ - \bar{\theta}^+\bar{\theta}) = \theta\theta^+ - \frac{1}{2} \text{Tr} \theta\theta^+$$

iso-scalar boson and baryons:

$$\Phi^0 = \frac{1}{2} \text{Tr}(\theta\theta^+)$$

is an isoscalar boson. Baryons are obtained by combining θ with Λ . The baryon gauge is given by $\Lambda \rightarrow \exp[ib]\Lambda$.

Parity

$$P_1: \quad \varepsilon_L \rightarrow \varepsilon_R, \quad \nu_L \rightarrow \nu_R \quad \text{for the leptons,}$$

$$P_2: \quad \Lambda_L \rightarrow \Lambda_R \quad \text{for the baryons.}$$

Under P_1 , θ transforms as

$$P_1\theta = -\theta^+$$

so that the roles played by transformations I and II are interchanged under P_1 . The groups I and II do not commute separately under parity, only the group with $R=Q$ does.

2. - Lack of parity conservation in strong interactions.

The fields θ are not parity eigen-states,

$$\theta^0 \rightarrow (\theta^0)^* ; \quad \theta^+ \rightarrow -(\theta^-)^* \quad \theta^- \rightarrow -(\theta^+)^* \quad \bar{\theta}^0 \rightarrow -(\bar{\theta}^0)^* .$$

Constructing the Φ we see that its components are not eigen-states of parity; we can find eigen-states of parity which do not form an iso-vector. This is the main result, due to the fact that the only rotation groups available (I and II) that can serve as the iso-group and hypercharge group, do not commute with parity. CP is good however.

The alternative scheme A' where the Majorana condition is imposed on ν , but not ε , has the same difficulty. The leptons work easily; but the scheme appears arbitrary. The paper will repay investigation as to the completeness of the solutions found. Of special interest is the question: do other rotation groups also exist?

3. - A generalization of the Pauli group (TOUSCHEK).

I want to discuss the symmetry properties of an assembly of n Majorana fields $\mu_i(x)$ ($i=1, \dots, n$). In the first part of the discussion I shall only consider the continuous symmetry groups which one can have for such an assembly; in the second part I would like to add some comments on GÜRSEY's work reported by PAULI.

The Majorana fields are defined by (*)

$$(1) \quad \mu_i^+(x) = \mu_i(x) ,$$

where the $^+$ indicate Hermitian conjugation and I shall consider a q -number

(*) I use Majorana's γ -gauge defined by $\gamma_r^+ = \gamma_r$; $\gamma_i^* = \gamma_i$ for ($i=1, 2, 3$) $\gamma_4^* = -\gamma_4$.

theory in which the canonical commutation relations are

$$(2) \quad \{\mu_i(x), \mu_k(x')\} = \frac{1}{2} \delta_{ik} \delta(x - x').$$

The symmetry group to be discussed is defined by the following requirements

(I) It commutes with the proper Lorentz group.

(II) It does not involve the coordinates.

(III) It is canonical, *i.e.* it leaves (2) invariant. The most general group which satisfies this requirement is

$$(3) \quad \mu' = \exp[A + iS\gamma_5]\mu.$$

The indices i, k have been suppressed. A is a real antisymmetric $n \times n$ matrix, and S is a real symmetrical $n \times n$ matrix. The requirement (I) has as a consequence that of all the Dirac γ -matrices only I and γ_5 occur; (II) is obviously satisfied and (III) is responsible for the fact that A must be antisymmetric and S symmetric. It follows immediately from (3) that the group has n^2 parameters.

For $n=1$ it reduces to the lepton gauge:

$$(4) \quad \mu' = \exp[i\gamma_5 S]\mu,$$

in which S is a real number.

For $n=2$ it is the Pauli group for a 4-component spinor. For if $\varrho_1 \varrho_2 \varrho_3$ are 3 Pauli matrices we have:

$$(5) \quad A = i\varrho_2\beta, \quad S = \delta I + \alpha\varrho_1 + \gamma\varrho_3.$$

The formal identity with the Pauli group is immediately seen if we put

$$(6) \quad \psi = \mu_1 - i\mu_2, \quad \psi^+ = \mu_1 + i\mu_2.$$

Then we have

$$\varrho_1\psi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} \mu_2 \\ \mu_1 \end{pmatrix} = \mu_2 - i\mu_1 = -i\psi^+$$

and similarly

$$\varrho_2\psi = \psi, \quad \varrho_3\psi = \psi^+.$$

Expressed in terms of 4-component spinors the group is then defined by

$$(7) \quad \psi' = \exp[i\delta\gamma_5] \cdot \exp[i\beta](a\psi + b\gamma_5\psi^+),$$

where $a = \cos a \cos \gamma - i \sin a \sin \gamma$ and $b = \sin a \cos \gamma + i \cos a \sin \gamma$ with obviously $a^2 + b^2 = 1$ and this is exactly the Pauli-group.

In the case $n = 4$ (which corresponds to the case of neutrons and protons, or also to the set of lepton spinors introduced in Gürsey's A-scheme) we have a group of 16 parameters and it is not likely to find a correspondingly high symmetry in nature.

This suggests that one should study subgroups of (3) defined by suitable supplementary conditions. I shall here consider the subgroup which satisfies postulates (I), (II), (III) and in addition allows the definition of a mass operator. I shall assume that the masses are equal. The subgroup then consists of all the members of (3) which leave

$$(8) \quad M = \mu \gamma_4 X \mu$$

invariant. The equality of masses then has the consequence that $X^2 = 1$ (i.e. that X has only eigenvalues ± 1). If $M \neq 0$ it follows from the commutation relations (2) that

$$(9) \quad X^T = X,$$

i.e. that X is a symmetrical matrix. For M to be Hermitian it must also be real. It is now immediately seen that (8) imposes the condition

$$(9) \quad [AX] = 0, \quad \{SX\} = 0.$$

It then follows that for $n = 1$ the group is empty. For $n = 2$ X can be chosen to be 1, or ϱ_1 or ϱ_3 . If $X = 1$ we remain with the gauge transformation $\psi' = \exp[i\beta]\psi$, for $X = \varrho_1$, $S = \varrho_3$ we remain with $\psi' = \psi \cos \gamma + i\gamma_5 \sin \gamma \psi^+$ with the single parameter γ and for $X = \varrho_3$ we have $S = \varrho_1$, which also leaves us with one parameter α .

The case $n = 4$ deserves special attention for two reasons: it corresponds to the neutron proton situation and it is the case which in Gürsey's discussion leads to the parity difficulty which has been explained in the previous talk.

In this case as in the case $n = 2$ we dispose of an algebra which allows us to give an explicit representation of the matrices A and S . Set $I, \Gamma_1, \dots, \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 = \Gamma_5$ be 16 Dirac matrices (Majorana gauge for simplicity). Then A must be a superposition with real coefficients of the 6 matrices

$$i\Gamma_4, \quad \Gamma_1 \Gamma_2, \quad \Gamma_2 \Gamma_3, \quad \Gamma_3 \Gamma_1, \quad \Gamma_4 \Gamma_5, \quad i\Gamma_5$$

and S a sum of the 10 matrices

$$I; \quad \Gamma_1, \Gamma_2, \Gamma_3, \quad i\Gamma_1 \Gamma_4, i\Gamma_3 \Gamma_4, i\Gamma_2 \Gamma_4; \quad i\Gamma_1 \Gamma_5, i\Gamma_2 \Gamma_5, i\Gamma_3 \Gamma_5.$$

The operator X must belong to the same set as S .

Let me first consider the case $X=1$. We then have $S=0$ and A arbitrary. The group has 6 Parameters and is isomorphic to the rotation group in 4-dimensions, which can be represented as the product of 2 commuting unitary groups, the generators of which are I_4 , I_5 , $-iI_4I_5$, for the one and $-iI_2I_3$, $-iI_3I_1$, $-iI_1I_2$ for the other. To make the argument more explicit and put ourselves into a position to give a physical interpretation for the case of, say, a neutron-proton system, we can use open products of two Pauli matrices ϱ and τ to define A . The most general form of A is then

$$(10) \quad A = A_1 + A_2,$$

with

$$(11) \quad \begin{cases} A_1 = i(\alpha\tau_1\varrho_2 + \beta\tau_2I + \gamma\tau_3\varrho_2), \\ A_2 = i(\alpha'\tau_2\varrho_1 + \beta'\varrho_2 + \lambda'\tau_2\varrho_3). \end{cases}$$

If we interpret the ϱ 's as the matrices with the properties $\varrho_1\Psi = -i\Psi^+$, $\varrho_2\Psi = \Psi$ and $\varrho_3\Psi = \Psi^+$ (applied to an 8 component spinor, we may write)

$$(12) \quad \exp[A_1]\Psi = \exp[i(\alpha\tau_1 + \beta\tau_2 + \gamma\tau_3)]\Psi$$

and the group A_1 can be interpreted as the isotopic spin group. The other group A_2 then has the following significance. Consider infinitesimal transformations, then

$$(13) \quad \delta\Psi = i\tau_2(-i\alpha' + \gamma')\Psi^+ + i\beta'\Psi.$$

The β' component is a gauge transformation under which neutrons and protons transform in the same way. The group A_2 bears the same relation to baryons conservation which A_1 bears to charge conservation.

We now show that this group does not allow the construction of isovectors, which are invariant under the proper Lorentz group. For, let

$$(14) \quad V_\alpha = \psi^+ \tau_\alpha \gamma \psi,$$

where γ is a combination of γ_4 and $\gamma_4\gamma_5$, be an isovector. Under (13) this goes into $V'_\alpha = V_\alpha + (-i\alpha' + \gamma')\psi^+ \tau_\alpha \tau_2 \psi + \text{hermitian conjugate}$; if we require $V'_\alpha = V_\alpha$ we must have $(\tau_\alpha \tau_2 \gamma)^T = (\tau_\alpha \tau_2 \gamma)$. But since $(\tau_\alpha \tau_2)^T = \tau_\alpha \tau_2$ it follows that $\gamma^T = \gamma$ but this is only true for vectors ($\gamma = \gamma_4 \gamma_\nu$) or tensors ($\gamma = \gamma_4 [\gamma_\mu \gamma_\nu]$) in ordinary 4-space.

As long as the condition $X^2=1$ is satisfied all the conclusions derived for $X=I$ hold for arbitrary X . This has been pointed out by J. S. BELL whose argument I follow.

We first observe that the transformation

$$(15) \quad \mu' = R\mu = \frac{1}{2}((1+x) + i\gamma_5(1-x))\mu$$

is a canonical transformation (though not necessarily belonging to the group (3), which does not contain reflections). R satisfies $R^+R=I$. It is now obvious that

$$(16) \quad \mu\gamma_4 X\mu = \mu'\gamma_4\mu':$$

it follows that if under the group $e^A\mu'$ transforms into $e^A\mu'$, μ must transform into $R^+e^AR\mu$. We therefore have a one to one correspondence between the « mass conserving » groups belonging to two arbitrary mass operators X_1 and X_2 provided that $X_1^2=X_2^2=1$.

This is of course in complete agreement with Gürsey's result, which appears as a special case of the theorem: In a theory with two basic 4-spinors which admits the formation of a scalar isoscalar it is impossible to form an isovector which transforms as a scalar under the proper Lorentz group.

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